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Automorphisms in Gauge Theories and the Definition of CP and P

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Abstract

We study the possibilities to define CP and parity in general gauge theories by utilizing the intimate connection of these discrete symmetries with the group of automorphisms of the gauge Lie algebra. Special emphasis is put on the scalar gauge interactions and the CP invariance of the Yukawa couplings.

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Contents

1 Introduction

The discrete transformations charge conjugation (C), parity (P) and CP played an important rôle in the development of particle physics. In particular, the hypothesis of P (C) non–conservation [1] and its subsequent experimental confirmation [2] and the discovery of CP violation [3] constituted both substantial progress in our understanding of weak interactions and gave incentives to important theoretical developments culminating in the advent of the Standard Model (SM) [4]. It is obvious, nowadays, that the suitable framework for discussing CP and P is given by spontaneously broken gauge theories with chiral fields as the basic fermionic degrees of freedom. In the SM, CP and P non–invariance appear in two totally different ways: parity is broken because left and right–handed fermionic multiplets belong to different representations of the gauge group whereas complex Yukawa couplings entail CP non–conservation through the Kobayashi–Maskawa mechanism [5] for three quark families. Thus the spontaneous breakdown of the gauge group $SU(2) \times U(1)$ only affects the manifestation of CP and P violation but has nothing to do with the non–invariance itself. On the other hand, on a more fundamental level, in an appropriate extension of the SM or in a Grand Unified Theory (GUT) [6] one might prefer to break CP or both CP and P spontaneously to provide the same footing for the breaking of continuous and discrete symmetries. This offers the possibility to relate the breaking of CP (and P) to the breaking of the gauge group and thus to have a more intimate relationship between CP (and P) non–conservation and the gauge structure in addition to the aesthetic point of view addressed to before.

Adopting the premise of spontaneous breakdown of CP and P we are immediately led to the question when and how CP and P transformations can be defined in a gauge theory before spontaneous symmetry breaking. This problem is particularly acute for P when left and right–handed irreducible fermionic representations do not match. It is the purpose of this paper to provide a thorough discussion of these questions, i.e. to consider the symmetry aspects leading to CP, P and C transformations in general gauge theories. Thereby we are motivated by the striking structural similarity between CP and P when formulated in a framework where all fermion fields have the same chirality. Such a formulation is always possible since it is equivalent, e.g., to the use of the right–handed field

$$(\chi_L)^c \equiv C\gamma_0^T \chi_L^* \tag{1.1}$$

instead of χ_L (for conventions and the definition of the matrix C, see app. A). In this work we will use right–handed fields for definiteness. It is well known that the key to an understanding of the relation between CP and P is given by the automorphisms of the gauge group G [7] and that a gauge theory containing only the gauge bosons and fermions is always CP invariant [7, 8]. These points will be worked out and commented upon in detail in the first chapters of this paper. Furthermore, gauge interactions of scalar fields, the second focus of this work, are dealt with in the same way and this part of the interaction is CP invariant as well. However, complications in the definition of CP and P transformations arise for Higgs fields in real and pseudoreal irreducible representations (irreps) of G. Finally, the third focus is on Yukawa interactions $\mathcal{L}_Y$. In this connection not

only the group structure of the couplings represented by Clebsch–Gordan coefficients is important but also the couplings pertaining to each group singlet contained in $\mathcal{L}_Y$. In the case of replication of irreps of G like the well–known families of quarks and leptons there is freedom of defining CP and P with respect to unitary rotations among the "families" (see refs. [9, 10] for the context of left–right symmetric models). Choosing one of these "horizontal rotations", e.g. in the definition of CP, and requiring CP invariance of $\mathcal{L}_Y$ we impose conditions on the Yukawa coupling constants. In the simplest case (when the horizontal rotations are proportional to unit matrices) and with appropriate phase conventions they have to be real . Thus the Yukawa interactions are not automatically CP invariant in contrast to the gauge interactions. In the discussions of the three main subjects of this work special attention will be paid to the existence of certain bases with respect to the group structure and to the horizontal structure to provide "canonical" forms of CP where its properties are especially transparent.

General remarks on the scope of the paper: To clarify the range of validity of this paper some remarks are in order. We will always deal with compact Lie groups G as gauge groups which entails unitary representations (reps) and the real Lie algebra $\mathcal{L}_c$ of G is compact (see app. B). On the other hand, requiring unitary reps on quantum theoretical grounds, it can be shown that one can confine oneself to compact groups [11]. We will not only cover simple groups but also groups of the type $G' \times G''$ and $G' \times U(1)$ with G' and G'' simple. In physical terms these cases require two independent gauge coupling constants. More complicated gauge structures like the SM gauge group can easily be discussed by an extension of these considerations. Special emphasis will be put on $G = G' \times G'$ (G' simple) extended by a discrete element such that only one gauge coupling constant is present. A typical case would be left–right symmetric models [12, 13].

In general there will also be symmetries of the Lagrangian which are not gauged, whether discrete or continuous. Therefore the total symmetry group will have the structure $G \times H$ where H is the group of these additional symmetries. Multiplets identical with respect to the gauge group may be distinguished by different transformation properties under H. In certain cases this can forbid operations to be discussed in what follows which involve non–trivial multiplicities of irreps of G. For instance, if in a model there is a complex scalar multiplet transforming according to a real irrep of G but to a complex irrep of H then this multiplet cannot be split into two real multiplets. We leave it with this caveat and concentrate on the gauge group from now on.

Although we will be mainly concerned with reps of $\mathcal{L}_c$ it will be tacitly assumed that these reps can be extended to reps of G. Furthermore, in our discussion of the various terms of the Lagrangian we will not cover the scalar potential because we think that in concrete cases it can be treated with the methods of this paper though its general discussion is involved. Finally, since for gauge theories the CPT theorem holds [14] (see also ref. [15]) and since CP is discussed extensively time reversal is only mentioned shortly in this work.

Plan of the paper: In sect. 2 the main features of CP and P are worked out in detail by means of three examples: QED, QCD and the $SO(10)$–GUT. CP–type transformations comprising CP and P are introduced in sect. 3 where Conditions A and B for invariance of the Lagrangian containing gauge fields and fermions are derived. Since Condition A requires that a CP–type transformation on the gauge fields corresponds to an automorphism of $\mathcal{L}_c$ sect. 4 is devoted to a detailed discussion of Aut $(\mathcal{L}_c)$. Sects. 5 and 6 treat Condition B. In sect. 5 CP transformations are introduced as special solutions of Condition B and the existence of the "CP basis" is discussed where all the generators of $\mathcal{L}_c$ are either symmetric or antisymmetric matrices. In sect. 6 the general solution of Condition B is expounded and parity transformations are defined by characteristic properties of the associated automorphisms. Several notions of CP–type transformations are introduced in sects. 5 and 6 to clarify the subject. They are related to each other in the following way:

$$\text{CP–type transformation} \left\{ \begin{array}{l} \text{generalized CP } \{ \text{ canonical CP} \\ \\ \text{parity} \left\{ \begin{array}{l} \text{internal parity} \\ \text{external parity} \end{array} \right. \end{array} \right.$$

The brace { signifies that the more general notion is found to the left of it. In the examples of sect. 2 on the one hand, the structural similarity between CP and P will become apparent, on the other hand, their distinctive features will be elucidated as well. QED and QCD provide examples of external parity whereas the $SO(10)$–GUT, where all fermionic degrees of freedom are in a 16–dimensional irrep, allows to define an internal parity. In sect. 7 the whole discussion is carried over to scalars with emphasis on the special cases of real and pseudoreal irreps. Sect. 8 is devoted to the invariance of the Yukawa couplings under CP–type transformations. Furthermore, the general solutions of the condition imposed on these couplings by a generalized CP transformation are derived. Transformations of the charge conjugation type are considered in sect. 9. Finally, in sect. 10 a summary is presented.

In app. A our conventions concerning the γ matrices and the charge conjugation matrix C can be found. All properties of Lie algebras necessary for our purposes are collected in app. B. In app. C spinor reps of $so(N)$ are defined via the Clifford algebra and app. D describes the Lie algebra isomorphisms $so(4) \cong su(2) \oplus su(2)$ and $so(6) \cong su(4)$. The remaining appendices E–I give proofs which are not carried out in the main body of the paper.

Notational remarks: For convenience we will often leave out the arguments x and $\widehat{x} = (x^0, -\vec{x})$ in CP–type transformations. Equivalence of reps will be denoted by $\sim$. The letter D will be used for reps of $\mathcal{L}_c$ and also of G. Its use should be clear from the context. Yet $-D^T$ refers only to the complex conjugate of the Lie algebra rep D. Finally let us collect the abbreviations used in this work: CSA, GUT, ON, SM, rep and irrep denote Cartan subalgebra, Grand Unified Theory, orthonormal, Standard Model, representation and irreducible representation, respectively.

2 Examples

Before coming to the general discussion it is quite instructive to consider some examples which enable us to get a feeling for the main features. In this light we will discuss now QED, QCD and the GUT based on the spinor rep of $SO(10)$ (strictly speaking, an irrep of $Spin(10)$, its universal covering group, see e.g. ref. [16]).

QED: Denoting the electron field by $e(x)$ we know that the Lagrangian is invariant under

$$\begin{array}{rrcl} CP: & e(x) & \rightarrow & -Ce(\widehat{x})^* \\ P: & e(x) & \rightarrow & \gamma^0 e(\widehat{x}) \end{array} \tag{2.1}$$

where $\widehat{x} = (x^0, -\vec{x})$. In terms of chiral fields eq. (2.1) reads

$$\begin{array}{rrcl} CP: & e_{L,R}(x) & \rightarrow & -Ce_{L,R}(\widehat{x})^* \\ P: & e_{L,R}(x) & \rightarrow & \gamma^0 e_{R,L}(\widehat{x}) \end{array} \tag{2.2}$$

expressing the fact that, having started with Dirac fields, CP does not mix chiralities whereas parity connects fields with different chirality. Defining

$$\chi_{R1} \equiv e_R, \qquad \chi_{R2} \equiv C\gamma_0^T e_L^* \tag{2.3}$$

we can get yet another view of eq. (2.1) [17], namely

$$\begin{array}{rcl} CP: \begin{pmatrix} \chi_{R1} \\ \chi_{R2} \end{pmatrix} & \rightarrow & \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} C \begin{pmatrix} \chi_{R1} \\ \chi_{R2} \end{pmatrix}^* \\ P: \begin{pmatrix} \chi_{R1} \\ \chi_{R2} \end{pmatrix} & \rightarrow & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} C \begin{pmatrix} \chi_{R1} \\ \chi_{R2} \end{pmatrix}^* . \end{array} \tag{2.4}$$

With

$$U_{CP} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad U_P \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.5}$$

we observe that the form of CP and P is only distinguished by the matrix U in the CP–type transformation

$$\begin{pmatrix} \chi_{R1} \\ \chi_{R2} \end{pmatrix} \rightarrow -UC \begin{pmatrix} \chi_{R1} \\ \chi_{R2} \end{pmatrix}^* . \tag{2.6}$$

How are U_{CP} and U_P characterized with respect to the gauge group $U(1)_{em}$? The chiral fields $\chi_{R1,2}$ transform with complex conjugate phases, i.e.

$$\chi_{R1} \rightarrow e^{i\alpha}\chi_{R1}, \qquad \chi_{R2} \rightarrow e^{-i\alpha}\chi_{R2}, \qquad e = \chi_{R1} + (\chi_{R2})^c \rightarrow e^{i\alpha} e \tag{2.7}$$

and the chiral field vector consists of two irreps which are exchanged under U_P whereas U_{CP} acts within the respective irreps. Thus, in the formulation where the previously

mentioned structural similarity becomes obvious, CP and P can be distinguished by group theoretical properties of the matrix U. Eq. (2.4) has to be supplemented by the transformation properties of the photon field

$$\begin{array}{rcl} CP: \quad A_\mu(x) & \rightarrow & -\varepsilon(\mu) A_\mu(\widehat{x}) \\ P: \quad A_\mu(x) & \rightarrow & \varepsilon(\mu) A_\mu(\widehat{x}) \end{array} \tag{2.8}$$

where the $\varepsilon(\mu)$ are the signs $+1$ for $\mu = 0$ and -1 for $\mu = 1, 2, 3$.

QCD: We confine ourselves to a single quark flavour q. It is well known that QCD is invariant under the transformations analogous to eq. (2.2) and with the same reasoning as before we get for the "CP–type" transformation

$$\chi_R \rightarrow -UC\chi_R^* \qquad \text{with} \qquad \chi_R = \begin{pmatrix} \chi_{R1} \\ \chi_{R2} \end{pmatrix} = \begin{pmatrix} q_R \\ (q_L)^c \end{pmatrix} \tag{2.9}$$

and

$$U_{CP} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \qquad U_P = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}. \tag{2.10}$$

Eq. (2.7) now reads

$$\chi_{R1} \rightarrow D\chi_{R1}, \qquad \chi_{R2} \rightarrow D^* \chi_{R2}, \qquad q = \chi_{R1} + (\chi_{R2})^c \rightarrow Dq \tag{2.11}$$

with $D \in SU(3)_c$ and eq. (2.8) is adapted to QCD by

$$\begin{array}{rcl} CP: W_\mu^a(x) & \rightarrow & \eta_a \varepsilon(\mu) W_\mu^a(\widehat{x}) \\ P: W_\mu^a(x) & \rightarrow & \varepsilon(\mu) W_\mu^a(\widehat{x}) \end{array} \tag{2.12}$$

with $a = 1, \ldots, 8$. The signs η_a depend on the transposition properties of the generators $\lambda_a/2$ where λ_a are the Gell–Mann matrices [18] and are obtained by

$$-\lambda_a^T = \eta_a \lambda_a. \tag{2.13}$$

As is well–known this choice of signs in eq.(2.12) also leaves invariant the pure gauge part of QCD. In sect. 3 (see also app. B) this fact will be connected with the mapping $\lambda_a \rightarrow \eta_a \lambda_a$ being an automorphism of the Lie algebra $su(3)$.

SO(10) – GUT: A more involved example of CP and P is provided by a Grand Unified Theory based on $SO(10)$ [19, 20, 21, 22] or actually on its universal covering group $Spin(10)$. Here, all 15 fundamental fermions of one SM family together with an additional right–handed neutrino carrying lepton number $+1$ fit nicely into a 16–dimensional spinor irrep. In order to implement CP and P a more elaborate analysis of the spinor irrep has to be performed. For our purpose, it is appropriate to take advantage of

$$so(10) \supset so(6) \oplus so(4) \cong su(4) \oplus su(2) \oplus su(2) \tag{2.14}$$

since the classification of the states according to $SU(4)_c \times SU(2)_L \times SU(2)_R$ is well known. Here, $SU(4)_c$ denotes the colour or Pati–Salam–$SU(4)$ [12] where lepton number is treated as "fourth colour". Furthermore, eq. (2.14) also represents a step towards a realistic pattern of spontaneous symmetry breaking. All necessary ingredients we are using in the following are explained in apps. C and D. Following refs. [20, 21], we denote the basis vectors of the 32–dimensional space $\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C}^2$ by

$$|s_1 s_2 s_3 s_4 s_5\rangle,$$

where $s_j = \pm 1$ or, abbreviated, $\pm$, and

$$|s_1 s_2 s_3 s_4 s_5\rangle = \mathbf{e}_{s_1} \otimes \mathbf{e}_{s_2} \otimes \mathbf{e}_{s_3} \otimes \mathbf{e}_{s_4} \otimes \mathbf{e}_{s_5}$$

with

$$\mathbf{e}_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.15}$$

The 16–dimensional subspaces for the spinor irreps $\{16\}$ and $\{\overline{16}\}$ are given by the two projectors $P_\pm$ on the 32–dimensional space where

$$P_\pm = \frac{\mathbf{1} \pm \Gamma_{11}}{2}, \quad \Gamma_{11} = \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3. \tag{2.16}$$

This means we are taking as basis vectors those with

$$\prod_{j=1}^{5} s_j = +1 \text{ or } -1 \tag{2.17}$$

for the $\{16\}$ or the $\{\overline{16}\}$, respectively.

The subalgebra $so(4)$ of $so(10)$ is assumed to be generated by M_{ij}, $1 \leq i < j \leq 4$, and $so(6)$ by M_{ij}, $5 \leq i < j \leq 10$. The CSA $\{F_3^c, Y_c, B-L\}$ of $su(4)_c$ is easily carried over to the spinor representation of $so(10)$ by using the Clifford algebra (see app. C) and the mapping of the Gell–Mann basis into $so(6)$ as given in app. D.[1] Thus we obtain

$$\begin{aligned}
F_3^c &= \frac{1}{2}\lambda_3 = \frac{1}{2}\text{diag }(1,-1,0,0) \\
&\quad \rightarrow (F_3^c)_s = \frac{1}{4}(-\mathbf{1} \otimes \mathbf{1} \otimes \sigma_3 + \mathbf{1} \otimes \sigma_3 \otimes \mathbf{1}) \otimes \mathbf{1}^{(2)} \\
Y_c &= \frac{1}{\sqrt{3}}\lambda_8 = \frac{1}{3}\text{diag }(1,1,-2,0) \\
&\quad \rightarrow (Y_c)_s = \frac{1}{6}(-\mathbf{1} \otimes \mathbf{1} \otimes \sigma_3 - \mathbf{1} \otimes \sigma_3 \otimes \mathbf{1} + 2\sigma_3 \otimes \mathbf{1} \otimes \mathbf{1}) \otimes \mathbf{1}^{(2)} \\
B-L &= \sqrt{2/3}\lambda_{15} = \frac{1}{3}\text{diag }(1,1,1,-3) \\
&\quad \rightarrow (B-L)_s = \frac{1}{3}(\mathbf{1} \otimes \mathbf{1} \otimes \sigma_3 + \mathbf{1} \otimes \sigma_3 \otimes \mathbf{1} + \sigma_3 \otimes \mathbf{1} \otimes \mathbf{1}) \otimes \mathbf{1}^{(2)}.
\end{aligned} \tag{2.18}$$

[1] The M_{ij} in eq. (D.4) have to be replaced by $M_{4+i\,4+j}$ to comply with the embedding of $so(6)$ into $so(10)$ chosen above.

Here, B and L are baryon and lepton number, respectively, and $\{F_3^c, Y_c\}$ generate the CSA of $su(3)_c$. The symbol $\mathbf{1}^{(p)}$ denotes the p-fold tensor product of the 2×2 unit matrix $\mathbf{1}$ and the subscript s refers to the spinor rep.

By a completely analogous procedure the two $su(2)$ subalgebras of $so(4)$ can be obtained:

$$\begin{aligned}
a_1 &= \frac{i}{4}\mathbf{1}^{(3)} \otimes (\sigma_2 \otimes \sigma_2 + \sigma_1 \otimes \sigma_1) \\
a_2 &= \frac{i}{4}\mathbf{1}^{(3)} \otimes (\sigma_2 \otimes \sigma_1 - \sigma_1 \otimes \sigma_2) \\
a_3 &= \frac{i}{4}\mathbf{1}^{(3)} \otimes (\mathbf{1} \otimes \sigma_3 - \sigma_3 \otimes \mathbf{1}) \\
b_1 &= -\frac{i}{4}\mathbf{1}^{(3)} \otimes (\sigma_2 \otimes \sigma_2 - \sigma_1 \otimes \sigma_1) \\
b_2 &= \frac{i}{4}\mathbf{1}^{(3)} \otimes (\sigma_2 \otimes \sigma_1 + \sigma_1 \otimes \sigma_2) \\
b_3 &= -\frac{i}{4}\mathbf{1}^{(3)} \otimes (\mathbf{1} \otimes \sigma_3 + \sigma_3 \otimes \mathbf{1}).
\end{aligned} \tag{2.19}$$

Comparing eq. (2.19) with eq. (D.1), we see the correspondences $A_j \leftrightarrow a_j$ and $-B_j^T \leftrightarrow b_j$. The advantage of the choice $\{-B_j^T\}$ compared with the equivalent $\{B_j\}$ will become clear at the end of this section. The algebra $\{a_j\}$ generates $SU(2)_L$ whereas $\{b_j\}$ generates $SU(2)_R$.

The remaining two elements of the Cartan subalgebra in the spinor rep of $so(10)$ are given by

$$\begin{aligned}
(I_{3L})_s = ia_3 &= \frac{1}{4}\mathbf{1}^{(3)} \otimes (\sigma_3 \otimes \mathbf{1} - \mathbf{1} \otimes \sigma_3) \\
(I_{3R})_s = ib_3 &= \frac{1}{4}\mathbf{1}^{(3)} \otimes (\sigma_3 \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_3).
\end{aligned} \tag{2.20}$$

This completes the definition of the CSA of $so(10)$ in the 32–dimensional spinor rep. We can now easily construct $SU(2)_L \times SU(2)_R$ multiplets by means of the lowering operators

$$\begin{aligned}
a_- &= i(a_1 - ia_2) = -\mathbf{1}^{(3)} \otimes \sigma_- \otimes \sigma_+ \\
b_- &= i(b_1 - ib_2) = -\mathbf{1}^{(3)} \otimes \sigma_- \otimes \sigma_-
\end{aligned} \tag{2.21}$$

with $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$ and find the basis vectors, omitting the $so(6)$ part,

$$\begin{aligned}
(2,1): &\quad |+-\rangle, -|-+\rangle \\
(1,2): &\quad |++\rangle, -|--\rangle.
\end{aligned} \tag{2.22}$$

Calling the quarks with (F_3^c, Y_c) quantum numbers $(1/2, 1/3)$, $(-1/2, 1/3)$, $(0, -2/3)$ red, yellow and blue, respectively, we can start with a right–handed red up–quark state $u_r = |++-++\rangle$. Then, applying b_- we obtain $d_r = -|++---\rangle$. The $su(4)$ multiplet

is completed by the raising and lowering operators

$$\begin{aligned}
\frac{1}{2}(\lambda_1 \pm i\lambda_2) &\rightarrow -\mathbf{1} \otimes \sigma_\pm \otimes \sigma_\mp \otimes \mathbf{1}^{(2)} \\
\frac{1}{2}(\lambda_4 \pm i\lambda_5) &\rightarrow \sigma_\pm \otimes \sigma_3 \otimes \sigma_\mp \otimes \mathbf{1}^{(2)} \\
\frac{1}{2}(\lambda_6 \pm i\lambda_7) &\rightarrow -\sigma_\pm \otimes \sigma_\mp \otimes \mathbf{1} \otimes \mathbf{1}^{(2)} \\
\frac{1}{2}(\lambda_9 \pm i\lambda_{10}) &\rightarrow \sigma_\pm \otimes \sigma_\pm \otimes \mathbf{1} \otimes \mathbf{1}^{(2)} \\
\frac{1}{2}(\lambda_{11} \pm i\lambda_{12}) &\rightarrow \sigma_\pm \otimes \sigma_3 \otimes \sigma_\pm \otimes \mathbf{1}^{(2)} \\
\frac{1}{2}(\lambda_{13} \pm i\lambda_{14}) &\rightarrow \mathbf{1} \otimes \sigma_\pm \otimes \sigma_\pm \otimes \mathbf{1}^{(2)}.
\end{aligned} \tag{2.23}$$

Thus, having embedded $SU(4)_c \times SU(2)_L \times SU(2)_R$ in $Spin(10)$ we find for its $(4,1,2)$ multiplet of right–handed quarks

$$\begin{pmatrix} u_r & u_y & u_b & N \\ d_r & d_y & d_b & e \end{pmatrix} =$$
$$\begin{pmatrix} |++-++\rangle & -|+-+++\rangle & |-++++\rangle & |---++\rangle \\ -|++---\rangle & |+-+--\rangle & -|-++--\rangle & -|-----\rangle \end{pmatrix}. \tag{2.24}$$

The electric charges of the states can easily be checked by means of the charge operator

$$\begin{aligned}
Q_{em} &= I_{3L} + I_{3R} + \frac{1}{2}(B-L) \rightarrow \\
(Q_{em})_s &= \frac{1}{2}\mathbf{1}^{(3)} \otimes \sigma_3 \otimes \mathbf{1} + \frac{1}{6}(\mathbf{1} \otimes \mathbf{1} \otimes \sigma_3 + \mathbf{1} \otimes \sigma_3 \otimes \mathbf{1} + \sigma_3 \otimes \mathbf{1} \otimes \mathbf{1}) \otimes \mathbf{1}^{(2)}.
\end{aligned} \tag{2.25}$$

From this and using F_3^c, Y_c (see eq. (2.18)) we find that N has the quantum numbers of a right–handed SM singlet neutrino, whereas e is the right–handed electron state. The state $|--+-+\rangle$ has the quantum numbers of an anti–red quark belonging to the (2,1) multiplet of $SU(2)_L \times SU(2)_R$ with eigenvalue $-1/2$ of $(I_{3L})_s$. This leads to the $(\bar{4},2,1)$ multiplets

$$\begin{pmatrix} u_r^c & u_y^c & u_b^c & \nu^c \\ d_r^c & d_y^c & d_b^c & e^c \end{pmatrix} =$$
$$\begin{pmatrix} -|--+-+\rangle & -|-+--+\rangle & -|+---+\rangle & |+++-+\rangle \\ -|--++-\rangle & -|-+-+-\rangle & -|+--+-\rangle & |++++-\rangle \end{pmatrix}. \tag{2.26}$$

The choice of signs in eq. (2.26) relative to eq. (2.24) will become clear from the discussion of parity. We see that with our states (2.24) and (2.26) we have projected out the space corresponding to $\prod_j s_j = -1$. All the quantum numbers correspond to the choice of right–handed fields in the $\{\overline{16}\}$ irrep. The construction of the gauge theory based on $so(10)$ and

the spinor representation $\{\overline{16}\}$ for the multiplet χ_R, the physical content of which has been described above, is now straightforward using the hermitian generators $T_{ij} = i\sigma_{ij}/2$, $(1 \leq i < j \leq 10)$ with $\sigma_{ij} = (M_{ij})_s$ (see eqs. (C.3) and (C.4)).

Anticipating sect. 3 and trying

$$CP : \chi_R(x) \to -C\chi_R(\widehat{x})^* \tag{2.27}$$

we merely have to check that invariance of the Lagrangian is achieved by an appropriate transformation of the gauge fields. In our case it is convenient to number the 45 gauge fields by $W_\mu^{ij}(x)$, corresponding to the generators T_{ij}. Analogously to the case of QCD it is easy to see that it is sufficient for CP invariance to transform the gauge fields as

$$CP : W_\mu^{ij}(x) \to \varepsilon(\mu)\eta_{ij}W_\mu^{ij}(\widehat{x}) \tag{2.28}$$

where the signs η_{ij} are obtained through

$$-\sigma_{ij}^T = \eta_{ij}\sigma_{ij}. \tag{2.29}$$

It is easy to calculate the η_{ij} recalling the definition of σ_{ij} in terms of the elements of the Clifford algebra (see app. C). We have

$$\Gamma_i^T = \xi_i\Gamma_i, \qquad \xi_i = (-1)^{i+1}, \tag{2.30}$$

therefore

$$-\sigma_{ij}^T = -\frac{1}{2}[\Gamma_i, \Gamma_j]^T = \xi_i\xi_j\sigma_{ij} \tag{2.31}$$

and

$$\eta_{ij} = (-1)^{i+j} = \xi_i\xi_j. \tag{2.32}$$

In sect. 5 it will be shown that one can always choose a basis in representation space such that $T_a^T = \pm T_a$ is valid for the generators. Eq. (2.29) tells us that

$$M_{ij} \to \eta_{ij}M_{ij}$$

is an automorphism of $so(10)$ (see app. B and sect. 3). We will see later that this guarantees a consistent choice of signs in eq. (2.28) such that the pure gauge part without fermions is also invariant under our CP transformation.

Turning to parity one is led to presume that this gauge theory is also parity invariant because the states (2.26) emerge from eq. (2.24) by a kind of "antiparticle formation". Whereas in QCD the states of $\{3\}$ and $\{\bar{3}\}$ correspond to each other, we now have such a correspondence within one $so(10)$ multiplet. In sect. 6 we will call the first kind of parity "external" and the second one "internal". One can indeed formulate a parity transformation by

$$P : \chi_R(x) \to -U_P C\chi_R(\widehat{x})^* \tag{2.33}$$

with

$$U_P = \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \mathbf{1} \tag{2.34}$$

and

$$P: W_\mu^{ij}(x) \rightarrow \varepsilon(\mu)\rho_{ij} W_\mu^{ij}(\widehat{x}) \tag{2.35}$$

where the signs ρ_{ij} are now obtained by (to be derived in sect. 3)

$$U_P(-\sigma_{ij}^T)U_P^{-1} = \rho_{ij}\sigma_{ij} \tag{2.36}$$

or else, using eqs. (2.31) and (2.32), by

$$U_P\sigma_{ij}U_P^{-1} = \eta_{ij}\rho_{ij}\sigma_{ij}. \tag{2.37}$$

Again, one can check that the pure gauge terms of the Lagrangian are invariant under the transformation (2.35). Note that the element σ_{12} of the CSA commutes with U_P, whereas σ_{34}, σ_{56}, σ_{78}, $\sigma_{9\,10}$ anticommute. In other words, U_P transforms eigenvectors of $(I_{3L} - I_{3R})_s$ into states with the same eigenvalue, whereas the eigenvalues of $(I_{3L} + I_{3R})_s$, $(B-L)_s$, $(F_3^c)_s$ and $(Y_c)_s$ are reversed.

Our sign conventions are such that U_P verifies

$$U_P \left(\begin{array}{cccc} u_r & u_y & u_b & N \\ d_r & d_y & d_b & e \end{array} \right) = \left(\begin{array}{cccc} u_r^c & u_y^c & u_b^c & \nu^c \\ d_r^c & d_y^c & d_b^c & e^c \end{array} \right) \tag{2.38}$$

where this notation means that U_P is applied to all the states in the parentheses. Taking into account eq. (2.19) and $\rho_{ij} = 1$ for $5 \leq i < j \leq 10$ we obtain

$$\begin{array}{rcll} U_P\sigma_{ij}U_P^{-1} & = & -\sigma_{ij}^T & \text{for } 5 \leq i < j \leq 10, \\ U_P a_j U_P^{-1} & = & -b_j^T, & \\ U_P b_j U_P^{-1} & = & -a_j^T. & \end{array} \tag{2.39}$$

From this it follows that the fields χ_{R1} and χ_{R2} corresponding to (4,1,2) and $(\overline{4}, 2, 1)$, respectively, in the irrep $\{\overline{16}\}$ transform with $SU(4)$ matrices which are exactly complex conjugate to each other. The same is true for the "diagonal $SU(2)$" generated by $\{a_j + b_j\}$. Thus χ_{R1} and $(\chi_{R2})^c$ transform alike under $SU(4)_c \times SU(2)_{\text{diag}}$ and correspond to e_R and e_L in QED, or q_R and q_L in QCD. Thus our conventions eqs. (2.19) and (2.38) were suggested to exhibit the common features of the three examples.

3 Conditions for CP–type transformations: gauge bosons and fermions

3.1 The pure gauge sector

To have a starting point for the discussion of CP–type transformations it is appropriate to repeat shortly the construction of gauge theories.

Let $\{T_a\}$ be the hermitian generators in the fermionic representation such that

$$[T_a, T_b] = i f_{abc} T_c \tag{3.1}$$

and

$$\text{Tr } (T_a T_b) = k\delta_{ab} \tag{3.2}$$

with totally antisymmetric coefficients f_{abc} (see app. B, first subsection). Defining

$$W_\mu \equiv T_a W_\mu^a \tag{3.3}$$

the pure gauge Lagrangian is given by (see, e.g., ref. [23])

$$\mathcal{L}_G = -\frac{1}{4k} \text{ Tr } (G_{\mu\nu} G^{\mu\nu}) \tag{3.4}$$

with

$$G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + ig[W_\mu, W_\nu] \equiv T_a G^a_{\mu\nu}. \tag{3.5}$$

Under a gauge transformation, the fermionic multiplet ω_R and the fields W_μ transform as

$$\begin{aligned} \omega_R(x) \quad &\rightarrow \quad U(x)\omega_R(x), & (3.6)\\ W_\mu(x) \quad &\rightarrow \quad U(x)W_\mu(x)U(x)^\dagger + \frac{i}{g}(\partial_\mu U(x))U(x)^\dagger & (3.7)\end{aligned}$$

with $U(x) = \exp\{-iT_a\alpha_a(x)\}$. As a result the field strength tensor $G_{\mu\nu}$ transforms according to the adjoint representation written as

$$G_{\mu\nu}(x) \rightarrow U(x)G_{\mu\nu}(x)U(x)^\dagger \tag{3.8}$$

therefore leaving $\mathcal{L}_G$ invariant.

Having fixed our notation we will now examine in detail the effect of a CP–type transformation in the gauge sector. The general form of such a transformation acting on the gauge boson multiplet is given by

$$W^a_\mu(x) \rightarrow \varepsilon(\mu)R_{ab}W^b_\mu(\widehat{x}) \quad \text{with } R \in O(n_G) \tag{3.9}$$

where n_G is the number of gauge bosons and thus equal to the number of group generators. The fields W^a_μ are real and therefore R is a real matrix. It will shortly become clear that the orthogonality condition (3.2) requires R to be orthogonal. Let us now consider the effect of the transformation (3.9) on the field strength tensor. For the terms with derivatives we have

$$\partial_\mu W^a_\nu(x) \rightarrow \varepsilon(\nu)R_{ad}\partial_\mu(W^d_\nu(\widehat{x})) = \varepsilon(\mu)\varepsilon(\nu)R_{ad}\widehat{\partial}_\mu W^d_\nu(\widehat{x}) \tag{3.10}$$

where $\widehat{\partial}_\mu$ is the derivative with respect to $\widehat{x}$. The commutator transforms according to

$$\begin{aligned} -gf_{abc}W^b_\mu(x)W^c_\nu(x) \quad &\rightarrow \quad -gf_{ab'c'}\varepsilon(\mu)\varepsilon(\nu)R_{b'b}R_{c'c}W^b_\mu(\widehat{x})W^c_\nu(\widehat{x}) \\ &= \quad -gR_{ad}f_{a'b'c'}R_{a'd}R_{b'b}R_{c'c}\varepsilon(\mu)\varepsilon(\nu)W^b_\mu(\widehat{x})W^c_\nu(\widehat{x}). & (3.11)\end{aligned}$$

Consequently, under a CP–type transformation $G^a_{\mu\nu}$ behaves as

$$G^a_{\mu\nu}(x) \rightarrow \varepsilon(\mu)\varepsilon(\nu)R_{ad}(\partial_\mu W^d_\nu - \partial_\nu W^d_\mu - g\widehat{f}_{dbc}W^b_\mu W^c_\nu)(\widehat{x}) \tag{3.12}$$

with

$$\widehat{f}_{dbc} = f_{a'b'c'} R_{a'd} R_{b'b} R_{c'c}. \tag{3.13}$$

This leads us to the first condition for invariance of $\mathcal{L}$ under a CP–type transformation:

$$\text{Condition A:} \quad f_{abc} = f_{a'b'c'} R_{a'a} R_{b'b} R_{c'c}. \tag{3.14}$$

In what follows eq. (3.14) will be referred to as Condition A. If it is fulfilled we get

$$G^a_{\mu\nu}(x) \to \varepsilon(\mu)\varepsilon(\nu) R_{ad} G^d_{\mu\nu}(\widehat{x}) \tag{3.15}$$

and $\int d^4x \;\; \mathcal{L}_G$ is clearly invariant under such a transformation. Note that eq. (3.10) together with eq. (3.2) already leads to R orthogonal in order to get invariance of the quadratic part of $\mathcal{L}_G$ under the transformation (3.9).

3.2 Fermions and gauge interactions

As mentioned before we choose to represent all the fermionic degrees of freedom by a single right–handed Weyl field vector ω_R transforming according to the rep $\{T_a\}$ (see app. A). Hence the fermionic Lagrangian is given by

$$\mathcal{L}_F = \overline{\omega_R} i \gamma^\mu (\partial_\mu + i g T_a {W_\mu}^a) \omega_R. \tag{3.16}$$

The general form of a CP transformation acting on the fermionic multiplet ω_R is given by

$$\omega_R(x) \to U \gamma^0 C \overline{\omega_R}(\widehat{x})^T = -U C \omega_R(\widehat{x})^* \tag{3.17}$$

where U, here, is a constant unitary matrix in representation space, i.e., in the same space on which the rep $\{T_a\}$ operates. It can be easily checked that the kinetic part of eq. (3.16) transforms as $\mathcal{L}_{F\text{kin}}(x) \to \mathcal{L}_{F\text{kin}}(\widehat{x})$ under eq. (3.17) whilst the interaction term leads to the invariance condition

$$-(U^\dagger T_b R_{ba} U)^T = T_a$$

which can readily be cast into the form

$$\text{Condition B:} \quad U(-{T_b}^T R_{ab}) U^\dagger = T_a. \tag{3.18}$$

In what follows we will refer to this equation as Condition B. It is easily verified that $\{-{T_b}^T R_{ab}\}$ fulfills the commutation relations (3.1) for R satisfying Condition A. This fact will be further exploited. Note that every CP–type transformation acting in a gauge theory with fermions can be described by a pair of matrices (R, U) defined above.

As we have seen R is an orthogonal $n_G \times n_G$ matrix and U a unitary matrix with the dimension of $\{T_a\}$. In general the rep $\{T_a\}$ will not be irreducible and we can therefore perform a decomposition into irreps

$$T_a = i \bigoplus_r (\mathbf{1}_{m_r} \otimes D_r(X_a)), \qquad \dim D_r = d_r \tag{3.19}$$

where $\{X_a\}$ is an ON basis of the real compact Lie algebra $\mathcal{L}_c$ (see app. B); m_r is the multiplicity of the irrep D_r in $\{T_a\}$ of dimension d_r; the direct sum runs over all irreps included in $\{T_a\}$ and its total dimension is $\sum_r m_r d_r$. We will call the degeneracy spaces "horizontal spaces" and the indices associated with them "horizontal indices". The decomposition (3.19) leads immediately to the following statement:

Theorem I: Let (R, U_0) be a solution of Conditions A and B and let $(R, U_1 U_0)$ be another solution. Then

$$U_1 = \bigoplus_r (u_r \otimes \mathbf{1}_{d_r}) \tag{3.20}$$

where the u_r are unitary $m_r \times m_r$ matrices.

Proof: Inserting $U_1 U_0$ into Condition B we obtain

$$T_a = U_1 [U_0 (-T_b{}^T R_{ab}) U_0^\dagger] U_1^\dagger = U_1 T_a U_1^\dagger .$$

Then Schur's lemma forces U_1 to be of the form (3.20). Since U_0 and $U_1 U_0$ are both unitary the matrices u_r are unitary as well. □

This simple theorem together with Theorem II of subsect. 6.1 will prove very useful in the discussion of solutions of Conditions A and B. In fact Theorem I shows that in order to solve Condition B one can concentrate on the determination of the group theoretical aspects of U with the freedom in the horizontal component simply given by eq. (3.20).

Defining a linear operator on $\mathcal{L}_c$ by (see app. B)

$$\psi_R : X_a \to R_{ba} X_b \tag{3.21}$$

we infer from Condition A that

$$[\psi_R(X_a), \psi_R(X_b)] = f_{abc} \psi_R(X_c). \tag{3.22}$$

Therefore ψ_R is an automorphism of $\mathcal{L}_c$ as well as $\psi_{R^{-1}}$ since the set of automorphisms of $\mathcal{L}_c$ forms a group (this can also be verified by examining Condition A). As a result Condition A can be formulated as

$$\psi_R \in \mathrm{Aut}\ (\mathcal{L}_c). \tag{3.23}$$

For every irrep D_r the complex conjugate irrep is given by $-D_r^T$. Clearly, $D_r \circ \psi_R$ defined by $(D_r \circ \psi_R)(X) \equiv D_r(\psi_R(X))$ is also an irrep. Thus Condition B can be read as

$$\bigoplus_r \left(\mathbf{1}_{m_r} \otimes (-D_r^T \circ \psi_{R^{-1}}) \right) \sim \bigoplus_r \left(\mathbf{1}_{m_r} \otimes D_r \right). \tag{3.24}$$

Eqs. (3.23) and (3.24) are purely Lie algebra theoretical conditions and will be discussed in sect. 4 and sects. 5 and 6, respectively.

Finally, let us write down the effect of changing the basis of the fermionic multiplet on a CP–type transformation. Let ω'_R be the field vector in the new basis and

$$\omega_R = Z\omega'_R. \tag{3.25}$$

As a result the new matrix U' in eq. (3.17) is given by

$$U' = Z^\dagger U Z^*. \tag{3.26}$$

It is important to note the complex conjugation on the right–hand side of eq. (3.26). This prevents the use of the well–known theorems for normal matrices (see subsect. 8.3 for a further discussion on basis transformations).

4 Automorphisms of $\mathcal{L}_c$

4.1 Types of automorphisms

The reformulation of Conditions A and B into eqs. (3.23) and (3.24), respectively, makes plain that the group of automorphisms of $\mathcal{L}_c$, Aut $(\mathcal{L}_c)$, plays an important rôle in our discussion. Therefore we want to set forth in this section all the details we will need in the following. The basic notions of Lie algebras necessary for this section can be found in app. B.

Inner and outer automorphisms [16, 24]: For each element Y of $\mathcal{L}_c$ we can define an automorphism ψ_Y of the form

$$\psi_Y(X) \equiv (\exp\ \text{ad}\ Y)(X) = e^Y X e^{-Y} \qquad \text{with} \qquad (\text{ad}\ Y)X \equiv [Y, X]. \tag{4.1}$$

Automorphisms of this kind are called inner. The representation of exp ad by exponentials, second equality of eq. (4.1), comes about due to the fact that all Lie algebras can be seen as matrix Lie algebras (Theorem of Ado [25]). It is clear from eq. (4.1) that the inner automorphisms of $\mathcal{L}_c$ from a group denoted by Int $(\mathcal{L}_c)$. For a connected Lie group G it is isomorphic to G/Z where Z is the centre of G [26]. Moreover, Int $(\mathcal{L}_c)$ is a normal subgroup of Aut $(\mathcal{L}_c)$ since if $\psi \in$ Aut $(\mathcal{L}_c)$ then $\psi(\text{ad}\ Y)\psi^{-1} = \text{ad}\ \psi(Y)$ and therefore $\psi(\exp\ \text{ad}\ Y)\psi^{-1} = \exp\ \text{ad}\ \psi(Y)$.

Outer automorphisms are automorphisms which are not inner. Clearly, it is sufficient to consider one representative of each coset in Aut $(\mathcal{L}_c)$/Int $(\mathcal{L}_c)$ in order to make a complete study of outer automorphisms.

Root rotations [16]: The group of root rotations Aut (Δ) is defined as a mapping of Δ, the set of (non–zero) roots of $\widetilde{\mathcal{L}}$ (complexification of $\mathcal{L}_c$), onto itself which fulfills

a) $\tau(\alpha + \beta) = \tau(\alpha) + \tau(\beta)\ \forall\ \alpha, \beta \in \Delta$ such that $\alpha + \beta \in \Delta$ and

b) $\tau(-\alpha) = -\tau(\alpha)$.

Aut (Δ) has an important normal subgroup, the Weyl group $\mathcal{W}$. It is the group generated by the elements S_α $(\alpha \in \Delta)$ which act on Δ as

$$S_\alpha \beta = \beta - 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha. \tag{4.2}$$

One can show that $S_\alpha \in$ Aut (Δ) $\forall\, \alpha \in \Delta$.

Associated with each root rotation $\tau \in$ Aut (Δ) there is an automorphism ψ_τ of the corresponding algebra $\widetilde{\mathcal{L}}$, which is also an automorphism of $\mathcal{L}_c$. The connection between Aut (Δ) and Aut $(\mathcal{L}_c)$ is given by the following theorem.

Theorem: For every $\tau \in$ Aut (Δ) there is a mapping ψ_τ of $\widetilde{\mathcal{L}}$ onto itself defined by

$$\psi_\tau(h_\alpha) = h_{\tau(\alpha)} \qquad \text{and} \qquad \psi_\tau(e_\alpha) = \chi_\alpha e_{\tau(\alpha)} \tag{4.3}$$

where $\chi_\alpha = \pm 1$ $\forall\, \alpha \in \Delta$ such that

$$\chi_\alpha = 1 \text{ for all simple roots,}$$

$$\chi_{\alpha+\beta} = \frac{N_{\tau(\alpha)\tau(\beta)}}{N_{\alpha\beta}}\chi_\alpha \chi_\beta \qquad \text{for } \alpha, \beta,\ \alpha+\beta \in \Delta^+ \tag{4.4}$$

$$\text{and } \chi_{-\alpha} = \chi_\alpha.$$

Then ψ_τ is an automorphism of $\widetilde{\mathcal{L}}$ and, moreover, restricted to $\mathcal{L}_c$ we also have $\psi_\tau \in$ Aut $(\mathcal{L}_c)$. See app. B for the notation in eqs. (4.3) and (4.4).

In what follows we will need the automorphism induced by root reflexion

$$\psi^\triangle \equiv \psi_{\tau_r} \qquad \text{with} \qquad \tau_r(\alpha) = -\alpha. \tag{4.5}$$

It is clear that $\tau_r \in$ Aut (Δ) and $\chi_\alpha = 1$ $\forall\, \alpha \in \Delta$. The index $\triangle$ stands for contragredient.

Another important class of root rotations in the case of simple Lie algebras is defined by those which are symmetries of the Dynkin diagram or, equivalently, of the Cartan matrix A_{jk} (see app. B). The group of these rotations is thus given by Aut $(DD) = \{\tau \in$ Aut $(\Delta)|\ A_{\tau(j)\tau(k)} = A_{jk}\ \forall\ j,k = 1,\ldots,\ell\}$. From the set of Dynkin diagrams depicted in fig. 1 it is clear that Aut (DD) can only be isomorphic to $\{e\}$ or $\mathbf{Z}_2$ or S_3. The latter case is only possible for D_4, the complexification of $so(8)$, where the three outer simple roots α_1, α_3, α_4 can be permutated leading to the permutation group S_3. For A_1, B_ℓ, C_ℓ and all exceptional algebras except E_6 there is no trivial diagram symmetry and therefore Aut (DD) is reduced to $\{e\}$. For A_ℓ $(\ell \geq 2)$ there is symmetry under the inversion of the order of the simple roots, i.e. $\tau(\alpha_j) = \alpha_{\ell+1-j}$, $j = 1, 2, \ldots, \ell$. For D_ℓ $(\ell \geq 5)$ exchange of $\alpha_{\ell-1}$ and α_ℓ is the unique Dynkin diagram symmetry, i.e., $\tau(\alpha_j) = \alpha_j$ $(j = 1, 2, \ldots, \ell-2)$, $\tau(\alpha_{\ell-1}) = \alpha_\ell$, $\tau(\alpha_\ell) = \alpha_{\ell-1}$. In E_6 we can reverse the order of the roots in the line containing the five roots $\alpha_1, \ldots, \alpha_5$, i.e., there the Dynkin diagram is symmetric under $\tau(\alpha_j) = \alpha_{6-j}$ $(j = 1, 2, \ldots, 5)$, $\tau(\alpha_6) = \alpha_6$. The automorphisms associated with non–trivial elements of Aut (DD) will be called diagram automorphisms and denoted by ψ_d.

The essence of the above discussion is related to the facts that for a simple Lie algebra $\mathcal{L}_c$, ψ_τ is an inner automorphism if and only if the root rotation τ is an element of the Weyl groyp $\mathcal{W}$ of $\widetilde{\mathcal{L}}$ and diagram automorphisms are always outer automorphisms so that we have [16, 24]

$$\text{Aut }(\mathcal{L}_c)/\text{Int }(\mathcal{L}_c) \cong \text{Aut }(\Delta)/\mathcal{W} \cong \text{Aut }(DD). \tag{4.6}$$

For our purposes it will be sufficient to use as representations of the cosets in Aut $(\mathcal{L}_c)$/Int $(\mathcal{L}_c)$ the automorphisms ψ_d, $\psi^\triangle$ or id (see table 1) depending on the algebra $\mathcal{L}_c$ under consideration. This covers the automorphism structure of simple Lie algebras.

4.2 Some examples of non–simple groups

We want to go a little beyond simple Lie algebras to include the most frequent cases occurring in model building. To determine Aut $(\mathcal{L}_c)$ in more complicated cases we need two trivial observations:

i) Let I be an ideal of $\mathcal{L}$ and $\psi \in$ Aut $(\mathcal{L})$. Then $\psi(I)$ is again an ideal.

ii) Let I_1, I_2 be ideals of $\mathcal{L}$ then also $I_1 \cap I_2$ is an ideal.

Let us now consider a few cases:

i) $\mathcal{L}_c = \mathcal{L}'_c \oplus \mathcal{L}''_c$ with $\mathcal{L}'_c$, $\mathcal{L}''_c$ simple and non–isomorphic:

Here we have

$$\text{Aut }(\mathcal{L}'_c \oplus \mathcal{L}''_c) \cong \text{Aut }(\mathcal{L}'_c) \times \text{Aut }(\mathcal{L}''_c). \tag{4.7}$$

Proof: $\mathcal{L}'_c \oplus 0$ is an ideal of $\mathcal{L}_c$. Then $(\mathcal{L}'_c \oplus 0) \cap \psi(\mathcal{L}'_c \oplus 0)$ is an ideal of $\mathcal{L}_c$ and also of $\mathcal{L}'_c$. Since $\mathcal{L}'_c$ is simple and $\mathcal{L}''_c \ncong \mathcal{L}'_c$ we have $\psi(\mathcal{L}'_c \oplus 0) = \mathcal{L}'_c \oplus 0$. The same reasoning is valid for the other summand $\mathcal{L}''_c$. □

In the following we will adopt the physical point of view that even when mathematically $\mathcal{L}'_c \cong \mathcal{L}''_c$, if the associated gauge couplings are different we will consider both Lie algebras as being different from each other.

ii) $\mathcal{L}_c = \mathcal{L}'_c \oplus \mathcal{L}'_c$, $\mathcal{L}'_c$ simple and associated to a group G:

In analogy with the previous case we have now

$$\text{Aut }(\mathcal{L}'_c \oplus \mathcal{L}'_c)/(\text{Aut }(\mathcal{L}'_c) \times \text{Aut }(\mathcal{L}'_c)) \cong \mathbf{Z}_2 \tag{4.8}$$

with the $\mathbf{Z}_2$ generated by

$$\psi_E((X,Y)) = (Y,X). \tag{4.9}$$

Now we can imagine that the Lie algebra is actually associated to a group G^* defined by enlarging $G \times G$ by an element E such that

$$G^* = (G \times G) \cup \{E\}, \qquad E^2 = (e,e), \qquad E(g,g')E = (g',g). \tag{4.10}$$

The physical background for this construction is that here the reps we are interested in are actually reps of G^* with the representation of E giving a symmetry reason for equal coupling constants for both group factors in G^*. In app. E the irreps of G^* are derived. Here we give the full list. Let D_r denote the irreps of G. Then all irreps of G^* are given by one of the following constructions:

$D_r^{\pm}$: $(g_1, g_2) \in G \times G$ is represented by $D_r(g_1) \otimes D_r(g_2)$ and $D(E)v \otimes w = \pm w \otimes v$.

$D_{rr'}(r \neq r')$: $(g_1, g_2) \in G \times G$ is represented by $(D_r(g_1) \otimes D_{r'}(g_2)) \oplus (D_{r'}(g_1) \otimes D_r(g_2))$ and $D(E)(v \otimes w, x \otimes y) = (y \otimes x, w \otimes v)$.

These irreps serve as a guideline for the construction of theories with gauge group $G \times G$ and equal coupling constants. In specific models ψ_E can be used as the automorphism associated with P (see sect. 6) or C (see sect. 9) and define in that way a transformation forcing equal gauge coupling constants.

iii) $\mathcal{L}_c = \mathcal{L}'_c \oplus u(1)$, $\mathcal{L}'_c$ simple:

Again one can show that

$$\text{Aut } (\mathcal{L}'_c \oplus u(1)) \cong \text{Aut } (\mathcal{L}'_c) \times \mathbf{Z}_2 \tag{4.11}$$

with $\mathbf{Z}_2$ generated by

$$\psi_u((X, X_u)) = (X, -X_u). \tag{4.12}$$

Actually, since $u(1)$ is abelian any transformation $X_u \to aX_u$ with $a \in \mathbf{R}\backslash\{0\}$ would be an automorphism but the gauge Lagrangian $\mathcal{L}_G$ can only be invariant under $a = \pm 1$ since $R_u = \text{diag }(1, \ldots, 1, \pm a)$ associated with ψ_u must be orthogonal.

4.3 Irreps and automorphisms

As discussed in subsect. 3.2 composition of irreps with automorphisms plays a crucial rôle in solving Condition B. Confining ourselves to simple Lie groups we know that any $\psi \in \text{Aut }(\mathcal{L}_c)$ can be written as $\psi = \psi_Y$ or $\psi_Y \circ \psi_d$. If D is an arbitrary rep then

$$\begin{aligned} D \circ \psi_Y &= e^{D(Y)} D e^{-D(Y)}, \\ D \circ \psi_Y \circ \psi_d &= e^{D(Y)} D \circ \psi_d e^{-D(Y)}. \end{aligned} \tag{4.13}$$

Therefore only outer automorphisms can give non–equivalent reps through composition.

To explore the effects of diagram automorphisms we consider first the action of $\tau \in \text{Aut }(DD)$ on the fundamental weights (see app. B):

$$\begin{aligned} \tau(\Lambda_j) &= \sum_{k=1}^{\ell} (A^{-1})_{jk} \tau(\alpha_k) = \sum_{k=1}^{\ell} (A^{-1})_{jk} \alpha_{\tau(k)} \\ &= \sum_{k=1}^{\ell} (A^{-1})_{\tau(j)\tau(k)} \alpha_{\tau(k)} = \sum_{k=1}^{\ell} (A^{-1})_{\tau(j)k} \alpha_k = \Lambda_{\tau(j)}. \end{aligned} \tag{4.14}$$

For convenience we did not distinguish between the diagram symmetry and its ensuing permutation of the indices $1,\ldots,\ell$, i.e. $\tau(\alpha_k)\equiv\alpha_{\tau(k)}$. Next we find the weights of the irrep $D\circ\psi_\tau$ by comparing it with D. Let $e(\lambda,q)$ be the eigenvector with the weight λ of D and $q=1,\ldots,m_\lambda$ where m_λ is the multiplicity of λ. Then

$$\begin{aligned} D(h_\alpha)e(\lambda,q) &= \lambda(h_\alpha)e(\lambda,q)=\langle\lambda,\alpha\rangle e(\lambda,q) \\ (D\circ\psi_\tau)(h_\alpha)e(\lambda,q) &= D(h_{\tau(\alpha)})e(\lambda,q)=\langle\lambda,\tau(\alpha)\rangle e(\lambda,q). \end{aligned} \tag{4.15}$$

Since τ is a root rotation one can show that [16]

$$\langle\tau(\beta),\tau(\gamma)\rangle=\langle\beta,\gamma\rangle \qquad \forall\,\beta,\gamma\in\Delta. \tag{4.16}$$

Therefore, if λ is a weight of D then $\tau^{-1}(\lambda)$ is a weight of $D\circ\psi_\tau$. This is valid for any root rotation τ. If in addition $\tau\in\mathrm{Aut}\,(DD)$ we obtain the highest weight of $D\circ\psi_\tau$ as (see eq. (4.14))

$$\tau^{-1}(\Lambda)=n_1\Lambda_{\tau^{-1}(1)}+\ldots+n_\ell\Lambda_{\tau^{-1}(\ell)}=n_{\tau(1)}\Lambda_1+\ldots+n_{\tau(\ell)}\Lambda_\ell \tag{4.17}$$

if $\Lambda=n_1\Lambda_1+\ldots+n_\ell\Lambda_\ell$ is the highest weight of D.

Now we can list the simple complex Lie algebras with non–trivial diagram symmetries and give the conditions for $D_\Lambda\sim D_\Lambda\circ\psi_d$ where we indicate the highest weight as a subscript. In all cases except D_4 the automorphism is unique. Thus we have $D_\Lambda\sim D_\Lambda\circ\psi_d$ if and only if

$$\begin{array}{lll} n_j=n_{\ell+1-j} & j=1,2,\ldots,\ell & \text{for } A_\ell\ (\ell\geq 2) \\ n_{\ell-1}=n_\ell & & \text{for } D_\ell\ (\ell\geq 5) \\ n_1=n_5,\ n_2=n_4 & & \text{for } E_6. \end{array} \tag{4.18}$$

For D_4 there are five non–trivial diagram automorphisms ψ_τ and $D_\Lambda\sim D_\Lambda\circ\psi_\tau$ only if

$$n_1=n_{\tau(1)}, \qquad n_3=n_{\tau(3)}, \qquad n_4=n_{\tau(4)}. \tag{4.19}$$

In subsect. 4.2 we have also defined the automorphism $\psi^\triangle$ associated with root reflexion. Clearly, $D_\Lambda\circ\psi^\triangle$ has the weight $-\lambda$ if λ is a weight of D_Λ. Therefore $D_\Lambda\circ\psi^\triangle\sim -D_\Lambda^T$ and $\psi^\triangle$ relates a rep to its contragredient rep. It is outer for A_ℓ $(\ell\geq 2)$, D_ℓ $(\ell=5,7,9,\ldots)$ and E_6. In these cases $D_\Lambda\circ\psi^\triangle\sim D_\Lambda\circ\psi_d$ and thus the condition for $D_\Lambda\sim -D_\Lambda^T$ is given by eq. (4.18). For all other simple Lie algebras $\psi^\triangle$ is inner and $D_\Lambda\sim -D_\Lambda^T$ for all their irreps (see table 1).

5 The existence of CP in gauge theories and Condition B

5.1 The automorphism associated with CP

In this section we will show that the fermionic and vector boson sectors of gauge theories (the terms $\mathcal{L}_F$ and $\mathcal{L}_G$ of the Lagrangian) are always CP symmetric. The same is true for

the couplings $\mathcal{L}_H$ of scalars with the gauge fields (see sect. 7). Imposing CP symmetry to the full Lagrangian $\mathcal{L}$ containing the terms $\mathcal{L}_G$, $\mathcal{L}_F$, $\mathcal{L}_H$, $\mathcal{L}_Y$, the Yukawa interactions, and $\mathcal{L}_S$, the scalar potential, will induce conditions on the only terms that may violate this symmetry — $\mathcal{L}_Y$ and $\mathcal{L}_S$. In sect. 8 we will discuss the conditions induced on $\mathcal{L}_Y$ — the simplest will be reality of Yukawa couplings but in general the situation is more complicated (see full classification in subsect. 8.3). We will not discuss invariance of $\mathcal{L}_S$ in this paper.

In our formulation a CP transformation of the fermionic and gauge boson multiplets is completely specified by the two matrices R and U defined in sect. 3. Requiring that CP reverses all quantum numbers of each field we see that R_{CP} corresponds to an automorphism which induces a reflexion of all the roots or, in other words, acts as $h \to -h$ on the CSA (as required by its physical interpretation). Hence we identify R_{CP} with the matrix representation of the contragredient automorphism defined by eq. (4.5) [7, 8]:

$$\psi_{CP} \equiv \psi^{\triangle}. \tag{5.1}$$

In what concerns the pure gauge term $\mathcal{L}_G$ of the Lagrangian we can conclude from the derivation of Condition A that it is always invariant under any automorphism of the Lie algebra irrespective of whether this is an inner or outer automorphism, contragredient or not.

One might think that identifying ψ_{CP} with $\psi^{\triangle}$ is too restrictive. This is, however, not the case because any two automorphisms that act in the same way on the CSA will only differ by an inner automorphism generated by an element of the CSA itself [24]. As we shall see in subsect. 6.1 using ψ_{CP} instead of $\psi^{\triangle}$ leads to equivalent conditions for CP invariance.

5.2 Canonical and generalized CP transformations

We consider now how to define a CP transformation on the fermion fields. In sect. 3, eq. (3.19) shows how the representation $\{T_a\}$ can be decomposed into irreps D_r. We know from the discussion of sect. 4 that

$$-\,D_r^T \circ \psi^{\triangle} \sim D_r \tag{5.2}$$

because the weights of both irreps are the same. This follows immediately from

$$\psi^{\triangle}(h_\alpha) = h_{-\alpha} = -h_\alpha. \tag{5.3}$$

Therefore there exist unitary matrices V_r such that

$$V_r(-D_r^T \circ \psi^{\triangle})V_r^{\dagger} = D_r \qquad \forall\, r. \tag{5.4}$$

Thus we define a "canonical CP transformation" by [7, 8]

$$(R^{\triangle}, U_0) \quad \text{with} \quad \psi_{R^{\triangle}} \equiv \psi^{\triangle}, \qquad U_0 = \bigoplus_r \mathbf{1}_{m_r} \otimes V_r. \tag{5.5}$$

Since $(R^\triangle, U_0)$ solves Conditions A and B we have a symmetry of $\mathcal{L}_G + \mathcal{L}_F$. This means that a gauge theory without Yukawa interactions and a scalar potential is automatically CP invariant (the scalar couplings to the gauge fields, $\mathcal{L}_H$, are treated like the fermionic couplings in sect. 7).

As already mentioned the physical reason to call the above transformation CP is that it turns around the signs of all members of the CSA in any rep. Thus it transforms fields into those with opposite quantum numbers. In order to see that this is indeed the case let $\mathcal{U}$ and $\mathcal{D}_r(g)$ be the operator implementations of CP and $D_r(g)$, respectively, on the Hilbert space of states and χ_R be a second quantized irreducible field multiplet. If CP is conserved we have

$$\mathcal{U}\chi_R(x)\mathcal{U}^{-1} = -V_r C \chi_R(\widehat{x})^* \quad \text{and} \quad \mathcal{D}_r(g)\chi_R(x)\mathcal{D}_r(g)^{-1} = D_r(g)\chi_R(x) \tag{5.6}$$

and, together with eq. (5.4), we obtain

$$\mathcal{D}_r(g)\mathcal{U}\chi_R(x)\mathcal{U}^{-1}\mathcal{D}_r(g)^{-1} = -V_r D_r(g)^* C\chi_R(\widehat{x})^* = (D_r \circ \psi^\triangle)(g)(-V_r C\chi_R(\widehat{x})^*). \tag{5.7}$$

As we know, in the irrep $D_r \circ \psi^\triangle$ the CSA is represented with opposite signs compared to D_r thus verifying the above statement.

Theorem I (sect. 3) tells us that given $R = R^\triangle$ the most general solution of Condition B is obtained by $U = U_1 U_0$ where U_1 is a horizontal transformation:

$$U_1 = \bigoplus_r u_r \otimes \mathbf{1}_{d_r}. \tag{5.8}$$

In the following we will call such transformations associated with $(R^\triangle, U_1 U_0)$ "generalized CP transformations" (see refs. [10, 27] for the context of left–right symmetric models and refs. [28, 29] for general considerations).

Considering the example of QCD in sect. 2 it is obvious that U_0 can be regarded as the identity matrix. Clearly this result can be generalized to the defining rep of $SU(N)$ for arbitrary N by generalizing the Gell–Mann matrices. The reason for $U_0 = \mathbf{1}$ is that these matrices are either symmetric or antisymmetric, therefore $-\lambda_a^T = \eta_a \lambda_a$ and the signs η_a are compensated by the transformation of the gauge bosons $W_\mu^a \to \eta_a W_\mu^a$. Thus we have $R^\triangle = \mathrm{diag}\,(\eta_1, \ldots, \eta_{n_G})$. It turns out that this is a general feature for any irrep of any semisimple compact Lie group if we choose the standard basis of $\mathcal{L}_c$ (see eq. (B.23)) and a suitable basis in representation space. To be specific, in app. F the following theorem is proved.

Theorem: For every irrep D of $\mathcal{L}_C$ there is an ON basis of $\mathbf{C}^d$ ($d = \dim D$) such that

$$D(X_a)^T = -\eta_a D(X_a), \qquad \eta_a^2 = 1 \qquad (a = 1, \ldots, n_G) \tag{5.9}$$

for the antihermitian generators of $\mathcal{L}_C$ in D. Furthermore the generators $\{X_a\}$ are those of the compact normal form of $\mathcal{L}_C$ and therefore the root reflexion $\psi^\triangle$ is given by

$$\psi^\triangle(X_a) = \eta_a X_a. \tag{5.10}$$

In other words, the matrices $D(X_a)$ are either imaginary and symmetric ($\eta_a = -1$) or real and antisymmetric ($\eta_a = 1$) and therefore a generalization of $-i\sigma_a/2$ (Pauli matrices) for $SU(2)$ and $-i\lambda_a/2$ (Gell–Mann matrices) for $SU(3)$ to arbitrary irreps of semisimple compact Lie algebras. The above basis will be called "CP basis" in the following. In this basis a canonical CP transformation, $(\mathrm{CP})_0$, can be simply represented by

$$(CP)_0 \to (R^\triangle, \mathbf{1}) \qquad \text{with } R^\triangle = \text{diag } (\eta_1, \ldots, \eta_{n_G}). \tag{5.11}$$

To be more explicit (see app. F) we have the following situation in a CP basis:

$$D(-iH_j) \quad (j = 1, \ldots, \ell), \quad D\left(\frac{e_\alpha - e_{-\alpha}}{i\sqrt{2}}\right) \quad (\alpha \in \Delta) : \eta_a = -1$$

(imaginary and symmetric)

$$D\left(\frac{e_\alpha + e_{-\alpha}}{\sqrt{2}}\right) \quad (\alpha \in \Delta) : \eta_a = 1 \tag{5.12}$$

(real and antisymmetric).

Therefore we get

$$\frac{1}{\sqrt{2}}\left[D\left(\frac{e_\alpha + e_{-\alpha}}{\sqrt{2}}\right) + iD\left(\frac{e_\alpha - e_{-\alpha}}{i\sqrt{2}}\right)\right] = D(e_\alpha) \qquad \text{real } \forall\ \alpha \in \Delta. \tag{5.13}$$

These results will be of importance when we consider the Yukawa couplings in sect. 8.

As a conclusion we have seen that a fermionic gauge theory (without scalars) is always CP invariant. The pure gauge term by itself is always invariant under any automorphism of the Lie algebra. In what concerns $\mathcal{L}_F$, the reason for invariance is due to the fact that the fermionic multiplet transforms in such a way that the CP transformed fermion fields are associated to the complex conjugate rep whilst the same transformation in the gauge boson sector will reverse this effect through $R^\triangle$ (this is obvious from the derivation of Condition B in sect. 3). This is the reason why invariance under a canonical CP transformation is verified irrespective of whether the automorphism is inner or outer. In all the examples of sect. 2, QED, QCD and the $SO(10)$–GUT, $\psi^\triangle$ is outer (see table 1).

6 On the general solution of Condition B

6.1 Introduction

In sect. 5 we discussed the solution of Condition B together with the requirement that R be such that it represents a CP transformation. Yet as we have seen in the examples of sect. 2 the form of CP and P transformations is the same in a formalism with fermion fields of a single chirality so that the difference between CP and P lies in the properties of R and U. This justifies that we study now Condition B with a general R. Therefore in this section we will discuss invariance of $\mathcal{L}_F$, the fermionic part of the Lagrangian of a gauge theory, under a CP–type transformation without constraining R to be the contragredient automorphism. The following theorem will simplify our discussion.

Theorem II: Let (R, U) be a pair of matrices that verify Conditions A and B and let R_I represent an inner automorphism of $\mathcal{L}_c$. Then the pairs

$$(R_I R, e^{iy_a T_a} U) \qquad \text{and} \qquad (R R_I, U(e^{iy_a T_a})^*) \tag{6.1}$$

where

$$\psi_{R_I}(X) = e^{-y_a X_a} X e^{y_a X_a} \tag{6.2}$$

are also solutions of Conditions A and B.

Proof: $R_I R$ and $R R_I$ are solutions of Condition A because the set of automorphisms form a group. Since ψ_{R_I} is inner there are real numbers y_a $(a = 1, \ldots, n_G)$ such that eq. (6.2) is fulfilled. Translating eq. (6.2) into the fermion representation we get

$$e^{iy_c T_c} T_a e^{-iy_c T_c} = R_{Iba} T_b. \tag{6.3}$$

Writing Condition B in terms of the second pair of eq. (6.1) we get

$$\begin{aligned} & U(e^{iy_c T_c})^* (-T_b^T (R R_I)_{ab}) (e^{-iy_c T_c})^* U^\dagger = \\ = \;\; & -U(e^{iy_c T_c} T_b e^{-iy_c T_c} (R R_I)_{ab})^* U^\dagger = \\ = \;\; & U(-T_c^T R_{Icb} (R R_I)_{ab}) U^\dagger = U(-T_b^T R_{ab}) U^\dagger = T_a. \end{aligned}$$

The other pair can be dealt with in an analogous way. □

Theorem II means for the discussion of invariance of $\mathcal{L}_F$ under CP–type transformations that if for a given automorphism there is a solution of Condition B (i.e., there is a matrix U that verifies the equality) then for any other automorphism differing from this one by an inner automorphism there will also be a solution of Condition B. The theorem also tells us how to relate the new matrix U to the initial one. In mathematical terms this means that it is only the quotient group Aut $(\mathcal{L}_c)$/Int $(\mathcal{L}_c)$ that needs to be considered so that we can confine ourselves to a particular representative of each coset.

Let us consider now the simple Lie algebras $\mathcal{L}_c$ in the light of Theorem II and distinguish three classes of Lie algebras (see table 1) [7]:

a) $\mathcal{L}_c = su(2)$, $so(2\ell+1)$ $(\ell \geq 2)$, $sp(2\ell)$ $(\ell \geq 3)$, cE_7, cE_8, cF_4, cG_2

Here no outer automorphisms exist and we take $\psi_R = id$ as representative. Since $\psi^\triangle$ is inner $-D^T \sim D$ is valid for all irreps D and therefore there is a matrix W such that

$$W(-D^T)W^\dagger = D. \tag{6.4}$$

Consequently Condition B is solvable without restrictions on the irrep content of $\{T_a\}$.

b) $\mathcal{L}_c = su(\ell+1)$ $(\ell \geq 2)$, $so(2\ell)$ $(\ell = 5, 7, 9, \ldots)$, cE_6

Here we have both inner and outer automorphisms. For the inner automorphisms we can consider again $\psi_R = id$ as representative, for the outer automorphisms we

can choose the contragredient one, $\psi_R = \psi^\triangle$, since in this case $\psi^\triangle$ is outer. In these Lie algebras any outer automorphism will be a composition of $\psi^\triangle$ with an inner automorphism

$\psi_R = id$: In general an irrep is not equivalent to its complex conjugate. As a result Condition B requires that for any irrep D also its complex conjugate $-D^T$ must be contained in $\{T_a\}$.

$\psi_R = \psi^\triangle$: Choosing a basis where the CSA in the irrep D is diagonal it is clear that the weights of $-D^T \circ \psi^\triangle$ are identical with those of D. Therefore $-D^T \circ \psi^\triangle \sim D$ and Condition B is solvable without restriction on $\{T_a\}$.

c) $\mathcal{L}_c = so(2\ell)$ ($\ell = 4, 6, 8, \ldots$)

In this case $\psi^\triangle$ is an inner automorphism, therefore all irreps D are equivalent to $-D^T$. Hence it is appropriate to represent Aut ($\mathcal{L}_c$) by the two cases $\psi_R = id$ and $\psi_R = \psi_d$, the unique diagram automorphism for $\ell = 6, 8, 10, \ldots$. As we know for $so(8)$ ($\ell = 4$) there are five non–trivial diagram automorphisms.

$\psi_R = id$: Analogously to case a) Condition B is solvable without restriction on $\{T_a\}$.

$\psi_R = \psi_d$: Condition B is solvable only if for any irrep D also $D \circ \psi_d$ is contained in $\{T_a\}$. The relationship between D and $D \circ \psi_d$ was discussed in subsect. 4.3.

We may also consider non–simple Lie groups in the light of Theorem II. For that purpose we take into account the results of sect. 4 and split the discussion into three generic classes.

i) $\mathcal{L}_c = \mathcal{L}'_c \oplus u(1)$, $\mathcal{L}'_c$ simple: We know that there is the automorphism $\psi_u : X_u \to -X_u$ eq. (4.12) on $u(1)$ in addition to Aut ($\mathcal{L}'_c$). Thus we can represent Aut ($\mathcal{L}_c$)/Int ($\mathcal{L}_c$) by $\psi = (\psi', id_u)$ or (ψ', ψ_u) where $\psi' \in \{id, \psi^\triangle, \psi_d\} \subset$ Aut ($\mathcal{L}'_c$). Each irrep D^u_k of $u(1)$ is characterized by the generator ik, $k \in \mathbf{Z}$. Denoting by D' an irrep of $\mathcal{L}'_c$ then the irreps of $\mathcal{L}_c$ are given by $D' \otimes D^u_k$. For $\psi = (\psi', id_u)$ also $(-D'^T \circ \psi') \otimes D^u_{-k}$ must be contained in $\{T_a\}$ whereas for $\psi = (\psi', \psi_u)$ it is sufficient to consider $\mathcal{L}'_c$.

ii) $\mathcal{L}_c = \mathcal{L}'_c \oplus \mathcal{L}''_c$, $\mathcal{L}'_c$, $\mathcal{L}''_c$ simple: This case leads to the discussion of each simple summand. Note that the irreps of $\mathcal{L}_c$ are given by the tensor products of the irreps of each summand [30] as for i). One should remember that here we have in mind different gauge coupling constants for $\mathcal{L}'_c$ and $\mathcal{L}''_c$ even for $\mathcal{L}'_c \cong \mathcal{L}''_c$ and therefore no automorphisms other than Aut ($\mathcal{L}'_c$) $\times$ Aut ($\mathcal{L}''_c$) exist.

iii) $\mathcal{L}_c = \mathcal{L}(G^*)$ with G^* defined in subsect. 4.2: The only additional automorphism is ψ_E given by eq. (4.9). It is easy to check that

$$D \circ \psi_E = D(E) D D(E) \tag{6.5}$$

for all irreps $D = D^\pm_r, D_{r,r'}$. Consequently ψ_E is as good as inner and one only has to consider a single summand.

6.2 On the definition of parity

In the definition of a CP transformation we had the automorphism $\psi^{\triangle}$ (changing the sign of all elements of the CSA $\mathcal{H}$) with $(\psi^{\triangle})^2 = id$. This suggests a mathematical definition of a parity transformation via an involutive automorphism ψ_P (i.e., verifying $\psi_P^2 = id$) which maps the CSA into itself and does not change the sign of the whole algebra. Therefore we can find an ON basis of $\mathcal{H}$ where [7]

$$\psi_P(H_j) = \mu_j H_j, \qquad \mu_j = \begin{cases} 1, & j = 1, \dots, p \\ -1, & j = p+1, \dots, \ell. \end{cases} \tag{6.6}$$

From the discussion in subsect. 6.1 it is obvious that if parity is a symmetry of the theory the matrix U_P associated with this transformation either maps a given irrep into itself or connects the irrep D to $(-D^T) \circ \psi_P \sim D \circ \psi^{\triangle} \circ \psi_P$. Both irreps have to be present in $\{T_a\}$ with the same multiplicity to fulfill Condition B. In the first case we call such a parity transformation *internal* with respect to D and in the second *external*. In sect. 2 we had the external cases of QED and QCD and the $\{\overline{16}\}$ of $so(10)$ as an internal case.

Let us for the time being consider the internal case with a single irrep D with dimension d. Then we can define a mapping:

$$\begin{array}{rcl} f : x & \rightarrow & U_P x^* \\ \mathbf{C}^d & \rightarrow & \mathbf{C}^d. \end{array} \tag{6.7}$$

Given an ON basis $e(\lambda, q)$ of weight vectors (see app. B) in the Hilbert space $\mathbf{C}^d$ associated with the irrep D, where $q = 1, \dots, m(\lambda)$ and $m(\lambda)$ denotes the multiplicity of λ, we have

$$D(H_j)[f(e(\lambda,q))] = -U_P D(\psi_P(H_j))^* e(\lambda,q)^* = -\mu_j \lambda(H_j) f(e(\lambda,q)). \tag{6.8}$$

To derive this relation we have used eq. (6.6) together with the fact that $D(H_j)$ is hermitian and that Condition B can be rewritten as

$$-\,U_P[D(\psi_P(iX))]^* U_P^\dagger = D(iX) \qquad \forall\, X \in \mathcal{L}_c. \tag{6.9}$$

As a result eq. (6.8) shows that f relates states characterized in general by different weights λ and λ_P given by

$$\begin{array}{rcl} \lambda & \leftrightarrow & (\lambda(H_1), \dots, \lambda(H_p), \lambda(H_{p+1}), \dots, \lambda(H_\ell)), \\ \lambda_P & \leftrightarrow & (-\lambda(H_1), \dots, -\lambda(H_p), \lambda(H_{p+1}), \dots, \lambda(H_\ell)). \end{array} \tag{6.10}$$

Thus weight vectors associated with λ are transformed into weight vectors associated with λ_P. Only vectors with $\lambda(H_1) = \dots = \lambda(H_p) = 0$ have no partner. Since $D(H_j)$ is hermitian $e(\lambda, q)$ is orthogonal to $e(\lambda_P, q)$ for $\lambda \neq \lambda_P$. Obviously the contragredient automorphism $\psi^{\triangle}$ does not establish such a relationship between weight vectors because for $\psi_P = \psi^{\triangle}$ we would have $\lambda = \lambda_P$ for all weights.

In the external case restricting oneself to single copies of D and $-D^T \circ \psi_P$, respectively, the above discussion goes through without change, but now $e(\lambda, q)$ and $e(\lambda_P, q)$ lie in

different spaces, namely in the spaces associated with D and $-D^T \circ \psi_P$, respectively. Taking into account non–trivial multiplicities in both cases, internal and external, does not alter the essence of our discussion but for simplicity we stick to single copies of irreps for the rest of this section.

We can make an analysis which has some resemblance to the one for CP in eqs. (5.6) and (5.7). As just before it will be valid for internal and external parity. Defining

$$\omega_R^{\lambda,q}(x) \equiv e(\lambda,q)^\dagger \omega_R(x) \tag{6.11}$$

we get

$$\mathcal{D}(e^{-isH_j})\omega_R^{\lambda,q}\mathcal{D}(e^{isH_j}) = e^{-is\lambda(H_j)}\omega_R^{\lambda,q}$$

and

$$\mathcal{D}(e^{-isH_j})\omega_R^{\lambda_P,q}\mathcal{D}(e^{isH_j}) = e^{-is\lambda_P(H_j)}\omega_R^{\lambda_P,q} \qquad \forall\ s \in \mathbf{R}. \tag{6.12}$$

This allows to define

$$\omega_L^{\lambda,q}(x) \equiv C\gamma_0^T(\omega_R^{\lambda_P,q}(x))^* \tag{6.13}$$

for $\lambda \neq \lambda_P$ which has the same quantum numbers $\lambda(H_j)$ $(j = 1,\ldots,p)$ as $\omega_R^{\lambda,q}(x)$. Then P transforms right into left and vice versa:

$$\begin{aligned}
\mathcal{U}_P\omega_R^{\lambda,q}(x)\mathcal{U}_P^{-1} &= -e(\lambda,q)^\dagger U_P C\omega_R(\widehat{x})^* \\
&= -(U_P^* e(\lambda,q))^\dagger U_P^* U_P C\omega_R(\widehat{x})^* = -e^{i\delta}C(\omega_R^{\lambda_P,q}(\widehat{x}))^* \\
&= e^{i\delta}\gamma_0\omega_L^{\lambda,q}(\widehat{x})
\end{aligned}$$

and, similarly,

$$\mathcal{U}_P\omega_L^{\lambda,q}(x)\mathcal{U}_P^{-1} = -\gamma_0\omega_R^{\lambda,q}(x). \tag{6.14}$$

To derive this equation we have taken into account eq. (6.8) and

$$U_P U_P^* = e^{-i\delta}\mathbf{1}_d \qquad \text{or} \qquad \begin{pmatrix} e^{i\delta}\mathbf{1}_d & 0 \\ 0 & e^{-i\delta}\mathbf{1}_d \end{pmatrix} \tag{6.15}$$

for internal and external parity, respectively[2]. Eq. (6.15) can be cast into a more general form:

If (R,U) is a CP–type transformation with $R^2 = \mathbf{1}$ then

$$[UU^*, T_a] = 0 \qquad \forall\ a = 1,\ldots,n_G.$$

Proof:

$$(UU^*)T_a(UU^*)^\dagger = U(UT_a^*U^\dagger)^*U^\dagger = -UT_b^*R_{ba}U^\dagger = U(-T_b^T R_{ab})U^\dagger = T_a.$$

□

In contrast to CP there is no canonical way to define P in general. With eq. (6.6) a mathematical definition was given but what is considered as parity in a model also

[2] δ is the phase of $U_P^*U_P$ in D or $-D^T \circ \psi_P$ for the internal or external case, respectively.

depends on the physical situation. A plausible requirement would be that the signs of the electromagnetic charge and the two colour charges are not changed under ψ_P. For a deeper connection of P and C with group theory the reader is referred to the discussion of charge conjugation in ref. [7] (see also sect. 9).

The external case of parity with $\psi_P = id$, $D \not\sim D^*$ is the type of parity we are used to (QED, QCD). In a basis where the matrices of D^* are just the complex conjugate matrices of D we simply form the Dirac field $\chi_D \equiv \chi_R + (\chi'_R)^c$ where χ_R and χ'_R transform according to D and D^*, respectively. In the $so(10)$ example (sect. 2) ψ_P is an inner automorphism.

7 CP–type transformations in the Higgs sector

7.1 Pseudoreal scalars and CP transformations

Considering a pseudoreal irrep D [16] there are two unitary matrices associated with it which are important in our context. The first one is given by the equivalence of the irrep and its complex conjugate. In Lie algebra form it is written as

$$W(-D^T)W^\dagger = D, \qquad W^T = -W \tag{7.1}$$

where antisymmetry of W follows from pseudoreality [16, 31]. The other unitary matrix is associated with the $(\mathrm{CP})_0$ symmetry:

$$U(-D^T \circ \psi^\triangle)U^\dagger = D. \tag{7.2}$$

The matrix U is symmetric:

$$U^T = U. \tag{7.3}$$

Proof: The symmetry of U is actually contained in the proof of app. F. There it is first demonstrated that $U^T = \pm U$. Then antisymmetry is excluded for irreps of semisimple compact Lie algebras. □

Eq. (7.3) is, of course, valid for arbitrary irreps. Clearly, in a CP basis $U \sim \mathbf{1}$ and part of the discussion here would be superfluous. However, we think that it is very instructive not to restrict our discussion to a CP basis but instead to have the general setting.

The matrix W allows to define

$$\widetilde{\phi} \equiv W\phi^* \qquad \text{with} \qquad \widetilde{\widetilde{\phi}} = -\phi \tag{7.4}$$

which transforms in exactly the same way as the scalar multiplet ϕ under the gauge group. It is more difficult to see that both fields also transform alike under CP:

$$CP : \phi \to U\phi^*, \qquad \widetilde{\phi} \to U\widetilde{\phi}^*. \tag{7.5}$$

This follows from the fact that W and U are related by

$$WU^\dagger W = -U \tag{7.6}$$

with an appropriate choice of phase, e.g., for W.

Proof of eq. (7.6): Putting $\psi^\triangle$ on the right–hand side of eq. (7.2) and using eq. (7.1) one obtains

$$U(-D^T)U^\dagger = (UW^\dagger)D(UW^\dagger)^\dagger = D \circ \psi^\triangle.$$

Taking advantage of $(\psi^\triangle)^2 = id$ one further derives

$$[(UW^\dagger)^2, D] = 0 \qquad \text{and thus} \qquad (UW^\dagger)^2 = \mu\mathbf{1}.$$

Choosing the phase of W such that $\mu = -1$ and making some algebraic manipulations one arrives at eq. (7.6). □

Proof of eq. (7.5): Under CP $\tilde{\phi}$ transforms as

$$\tilde{\phi} \to WU^*\phi = WU^*W^T\tilde{\phi}^* = -WU^\dagger W\tilde{\phi}^* = U\tilde{\phi}^*$$

where in the last step eq. (7.6) has been taken into account. □

Let us assume now that there are m copies $\phi_1, \ldots, \phi_m$ all in the same irrep D. Putting the m multiplets into a vector χ then $\mathcal{L}_H$, the scalar kinetic and gauge Lagrangian, can be rewritten as [9, 10]

$$\mathcal{L}_H(\chi) = \frac{1}{2}\mathcal{L}_H(\chi) + \frac{1}{2}\mathcal{L}_H(\tilde{\chi}). \tag{7.7}$$

This allows for a wider class of generalized CP transformations for pseudoreal scalars leaving $\mathcal{L}_H$ invariant defined by

$$CP : \begin{pmatrix} \chi \\ \tilde{\chi} \end{pmatrix} \to HU \begin{pmatrix} \chi \\ \tilde{\chi} \end{pmatrix}^* \tag{7.8}$$

where H is a unitary "horizontal" $2m \times 2m$ matrix. One has to take into account, however, that $\tilde{\chi}$ is related to χ via definition (7.4) which gives a restriction on H. With

$$H = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{7.9}$$

this restriction is derived by the requirement

$$W(AU\chi^* + BU\tilde{\chi}^*)^* = CU\chi^* + DU\tilde{\chi}^*. \tag{7.10}$$

Exploiting this condition by using eq. (7.6) in the form $WU^* = UW^*$ we finally obtain $D = A^*$, $C = -B^*$ and therefore

$$H = \begin{pmatrix} A & B \\ -B^* & A^* \end{pmatrix} \in Sp(2m) \tag{7.11}$$

where $Sp(2m) = \{H \in U(2m) |\, H^T J_m H = J_m\}$ is the unitary symplectic group [16]. J_m is defined by

$$J_m = \begin{pmatrix} 0 & \mathbf{1}_m \\ -\mathbf{1}_m & 0 \end{pmatrix}.$$

In a CP basis one can say more about the matrices U and W. With the phase convention of eq. (7.6) we get the following result:

$$\text{CP basis} \Rightarrow U = \mathbf{1}, \qquad W = W^* = -W^T \tag{7.12}$$

where $U = \mathbf{1}$ is the convention of the $(\mathrm{CP})_0$ transformation in the CP basis.

7.2 Real scalars in complex disguise and CP transformations

Scalar multiplets belonging to potentially real irreps D [16] can also be represented by complex fields. In general, in a CP basis, matrices of a potentially real irrep are complex. When these matrices are rotated into a real rep the scalar multiplets are also transformed and there are two possibilities – either the scalar multiplets also become real or else they remain complex in which case they can be split into two real multiplets belonging to the same real irrep. The second case will be illustrated by a familiar example in the gauge theory $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$. The treatment of two real multiplets collected in a complex multiplet ϕ can proceed along the same lines as in the case of pseudoreal scalars by the introduction of the corresponding field $\widetilde{\phi}$, instead of splitting it into its real components. When ϕ represents a single real irrep it coincides with $\widetilde{\phi}$ in the real basis so that its introduction in this case would be meaningless.

For real scalars one has now three unitary matrices associated with the irrep. U is given as in eq. (7.2), W is now symmetric [16, 31]

$$W(-D^T)W^\dagger = D, \qquad W^T = W \tag{7.13}$$

and V transforms D_R, an explicitly real realization of the potentially real rep D, into D:

$$D = VD_RV^\dagger. \tag{7.14}$$

Let us first have a look on the relationship between the $(\mathrm{CP})_0$ transformation in D_R and D. To do this we note that

$$WU^\dagger W = U \tag{7.15}$$

and

$$V^\dagger W V^* = \mathbf{1} \tag{7.16}$$

with appropriate phase factors for W and V.

Proof: Eq. (7.15) is proved as before in the pseudoreal case but now we choose a phase of W such that we have a plus sign instead of minus in eq. (7.6). The reason for this choice will become clear when we later go to a CP basis. The second relation derives from inserting eq. (7.14) into eq. (7.13) giving

$$WV^*D_RV^TW^\dagger = VD_RV^\dagger$$

or

$$[V^\dagger W V^*, D_R] = 0.$$

With Schur's lemma and a phase choice for V we obtain eq. (7.16). □

Let $\varphi = V^\dagger\phi$ be the real scalar field. Then the CP transformation on φ is given by

$$CP : \varphi \to V^\dagger U V^* \varphi \tag{7.17}$$

and

$$V^\dagger U V^* \text{ real}. \tag{7.18}$$

Proof: To prove eq. (7.18) we rewrite eq. (7.15) as

$$U^* W = W^* U \qquad \text{or} \qquad U^* V V^T = V^* V^\dagger U.$$

Therefore

$$(V^\dagger U V^*)^* = V^T (U^* V V^T) V^* = V^T (V^* V^\dagger U) V^* = V^\dagger U V^*.$$

□

Thus the general CP formalism is consistent with real fields. Choosing the CP basis we now have

$$\text{CP basis} \Rightarrow U = \mathbf{1}, \qquad W = W^* = W^T. \tag{7.19}$$

As before W is real. This is a consequence of the phase choice in eq. (7.15).

Now we come to the second topic, namely to the discussion of a complex multiplet comprising two real multiplets transforming under the same irrep. As in the pseudoreal case we define

$$\widetilde{\phi} \equiv W \phi^* \qquad \text{with} \qquad \widetilde{\widetilde{\phi}} = \phi. \tag{7.20}$$

Note that now we have a plus sign in the second relation (compare to eq. (7.4)) because $WW^* = \mathbf{1}$ for real irreps. As before ϕ and $\widetilde{\phi}$ transform alike under CP, eq. (7.5). If D has multiplicity m we can form the vectors χ, $\widetilde{\chi}$. Then eq. (7.7) is valid and CP can be defined as in eq. (7.8). But now the restriction on H differs from eq. (7.11). Imposing the same condition as in the pseudoreal case (see eq. (7.10)) but taking into account the properties of U and W in the real case we obtain [10]

$$H = \begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix}. \tag{7.21}$$

It is easy to check that the matrices H of the form eq. (7.21) are exactly those unitary matrices for which

$$H^T I H = I \qquad \text{with} \qquad I = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \tag{7.22}$$

is valid. Therefore they form a group. This group is identical with $O(2m)$ up to the basis transformation

$$Z^\dagger H Z = H' \in O(2m) \qquad \text{with} \qquad Z = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1} & i\mathbf{1} \\ \mathbf{1} & -i\mathbf{1} \end{pmatrix}. \tag{7.23}$$

This can readily be seen by inserting eq. (7.23) into eq. (7.22) where we obtain

$$H'^T (Z^T I Z) H' = Z^T I Z = \mathbf{1}_{2m}. \tag{7.24}$$

Defining real fields χ_1, χ_2 via

$$V^\dagger \chi = \frac{\chi_1 + i\chi_2}{\sqrt{2}} \tag{7.25}$$

the CP transformation (7.8) for the potentially real irrep D is now of the form

$$CP: \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \to (Z^\dagger H Z)(V^\dagger U V^*) \begin{pmatrix} \chi_1 \\ -\chi_2 \end{pmatrix} \tag{7.26}$$

after some calculation. Both matrix products embraced by the parentheses are real. This again shows the consistency of the formalism. Clearly, both ϕ and $\widetilde{\phi}$ have to be coupled in the Yukawa sector to get the most general interaction.

It may look strange that real scalars are packed together in complex fields. But we will see now by an example from $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ [13, 32, 33] that such cases are not uncommon. There a multiplet ϕ_m transforming as (2,2,0) exists which gives masses to the quarks. It is commonly written as a 2×2 matrix of complex fields and transforms as

$$\phi_m = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \to U_L \phi_m U_R^\dagger, \qquad U_{L,R} \in SU(2) \tag{7.27}$$

under the gauge group. The index m denotes the 2×2 matrix version of the field. $\widetilde{\phi}_m$ is given by

$$\widetilde{\phi}_m = \tau_2 \phi_m^* \tau_2 = \begin{pmatrix} \phi_{22}^* & -\phi_{21}^* \\ -\phi_{12}^* & \phi_{11}^* \end{pmatrix} \tag{7.28}$$

where τ_2 is the second Pauli matrix. Switching to a vector notation we read off W from eq. (7.27):

$$\phi \equiv \begin{pmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{12} \\ \phi_{22} \end{pmatrix}, \qquad \widetilde{\phi} = W\phi^* \qquad \text{with} \qquad W = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{7.29}$$

Since $W^T = W$ we suspect immediately that (2,2,0) is a real rep. This can indeed be confirmed by recalling the following theorem.

Theorem: The irreps of a direct product of groups $G \times G'$ are exactly the tensor products of irreps D and D' of G and G', respectively [30]. If D and D' are both real or both pseudoreal then $D \otimes D'$ is real [34].

In our case the defining irrep of $SU(2)$ is pseudoreal having spin $j = 1/2$ and therefore the rep (7.27) actually decays into two real irreps. Since this is an instructive example we want to show this explicitly.

With

$$U_L = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \quad |a|^2 + |b|^2 = 1 \quad \text{and} \quad U_R = \begin{pmatrix} c & d \\ -d^* & c^* \end{pmatrix}, \quad |c|^2 + |d|^2 = 1 \tag{7.30}$$

we find the corresponding transformations for the vector ϕ

$$U_L \phi_m \quad \to \quad \begin{pmatrix} U_L & 0 \\ 0 & U_L \end{pmatrix} \phi \equiv \widehat{U}_L \phi,$$

$$\phi_m U_R^\dagger \quad \to \quad \begin{pmatrix} c^* & 0 & d^* & 0 \\ 0 & c^* & 0 & d^* \\ -d & 0 & c & 0 \\ 0 & -d & 0 & c \end{pmatrix} \phi \equiv \widehat{U}_R \phi. \tag{7.31}$$

Of course, now we have $W \widehat{U}^*_{L,R} W^\dagger = \widehat{U}_{L,R}$. If we perform a basis transformation we get

$$\phi = Z\phi' \Rightarrow \widehat{U}'_{L,R} = Z^\dagger \widehat{U}_{L,R} Z \qquad \text{and} \qquad W' = Z^\dagger W Z^*. \tag{7.32}$$

The last relation can easily be obtained from eq. (7.13). It is obvious that in the basis where $W' = \mathbf{1}$ we have real representation matrices. With

$$V \equiv Z = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 0 & -i & 1 \\ 0 & 0 & -i & -1 \\ 1 & -i & 0 & 0 \end{pmatrix} \tag{7.33}$$

we have indeed $W' = \mathbf{1}$ and

$$\begin{aligned} \widehat{U}'_L \quad &\to \quad \begin{pmatrix} a_1 & -a_2 & b_2 & b_1 \\ a_2 & a_1 & -b_1 & b_2 \\ -b_2 & b_1 & a_1 & a_2 \\ -b_1 & -b_2 & -a_2 & a_1 \end{pmatrix}, \qquad \begin{array}{l} a_1 = \mathrm{Re}\, a, \quad a_2 = \mathrm{Im}\, a \\ b_1 = \mathrm{Re}\, b, \quad b_2 = \mathrm{Im}\, b \end{array} \\ \widehat{U}'_R \quad &\to \quad \begin{pmatrix} c_1 & c_2 & -d_2 & -d_1 \\ -c_2 & c_1 & -d_1 & d_2 \\ d_2 & d_1 & c_1 & c_2 \\ d_1 & -d_2 & -c_2 & c_1 \end{pmatrix}, \qquad \begin{array}{l} c_1 = \mathrm{Re}\, c, \quad c_2 = \mathrm{Im}\, c \\ d_1 = \mathrm{Re}\, d, \quad d_2 = \mathrm{Im}\, d. \end{array} \end{aligned} \tag{7.34}$$

Clearly, the new field multiplet ϕ' ($\widetilde{\phi}' = \phi'^*$) is still complex and we can split it into two real multiplets as in eq. (7.25).

This concludes our discussion of real scalars. The left–right symmetric example will be taken up once more in sect. 9 where CP will be discussed together with C and P in this context.

7.3 The general case

In the general case the scalar multiplets may contain real, pseudoreal and complex irreps. We put them together into a vector

$$\Phi = \begin{pmatrix} \phi_R \\ \phi_P \\ \widetilde{\phi}_P \\ \phi_C \\ \phi_C^* \end{pmatrix} \qquad \text{and} \qquad D_\mu \Phi = (\partial_\mu + ig\mathcal{T}_a W^a_\mu)\Phi \tag{7.35}$$

where the indices R, P and C denote real, pseudoreal and complex, respectively. D_μ is the covariant derivative. Therefore we can write the scalar kinetic and gauge Lagrangian as

$$\mathcal{L}_H = \frac{1}{2}(D_\mu \Phi)^\dagger (D^\mu \Phi). \tag{7.36}$$

In this way we summarize the gauge interactions into one formula with automatically correct factors 1/2 for ϕ_R and 1 for ϕ_P and ϕ_C since the contribution of ϕ_P and $\widetilde{\phi}_P$ are equal in $\mathcal{L}_H$ (see eq. (7.7)) and the same holds true for ϕ_C and ϕ_C^* (of course, if in the ϕ_C sector $\mathcal{T}_a$ is given by $\mathcal{T}_a^C$ then for ϕ_C^* it has to be $-(\mathcal{T}_a^C)^T$).

In analogy to the fermionic sector we define a CP–type transformation by

$$\Phi(x) \to U_H \Phi(\widehat{x})^* \tag{7.37}$$

leading to

$$\text{Condition B}_\text{H}: \quad U_H(-\mathcal{T}_b^T R_{ab}) U_H^\dagger = \mathcal{T}_a. \tag{7.38}$$

Clearly, Condition $\mathrm{B_H}$ is connected to Condition B by the same automorphism represented by R. For the total Lagrangian a CP–type transformation is characterized by the triple (R, U, U_H).

It is clear from eq. (7.37) that it is always possible to define a CP transformation in $\mathcal{L}_H$ in the same way as is done in the fermionic sector. Therefore as for CP–type transformations it remains to consider the automorphisms $\psi_R = id$ and $\psi_R = \psi_d$ in the relevant cases.

When considering the algebras $su(\ell+1)$ ($\ell \geq 2$), $so(2\ell)$ ($\ell = 5, 7, 9, \ldots$) and cE_6 in the general discussion of a CP–type transformation the automorphism $\psi_R = id$ is relevant in addition to $\psi^\triangle$. Obviously, if we consider the scalars ϕ_C in complex irreps we automatically have their complex conjugate irreps in in $\mathcal{L}_H$ eq. (7.35) so that Condition $\mathrm{B_H}$ can be solved for $\psi_R = id$ and complex reps. For real irreps and $\psi_R = id$ Condition $\mathrm{B_H}$ becomes trivial because in this case $-D^T = D$ in a basis where the fields are real. In the case of pseudoreal irreps and $\psi_R = id$ the CP–type transformation has to be defined as $\phi \to W\phi^*$ with W specified by eq. (7.1). A simple calculation reveals that now eqs. (7.5), (7.8) and (7.11) are valid with U replaced by W. This shows that for all simple algebras except $so(2\ell)$ with $\ell = 4, 6, 8, \ldots$ (see table 1) arbitrary CP–type transformations are always symmetries of $\mathcal{L}_H$ and in this sense the existence of a CP–type symmetry for $\mathcal{L}_H$ is more likely than for $\mathcal{L}_F$.

For the simple Lie algebras $so(2\ell)$ with $\ell = 4, 6, 8, \ldots$ the relevant case is $\psi_R = \psi_d$, a diagram automorphism. In this case we have $(-D_\Lambda^T) \circ \psi_d \sim D_\Lambda \circ \psi_d \not\sim D_\Lambda$ if and only if $n_{\ell-1} \neq n_\ell$ for the highest weight Λ ($\ell \geq 6$) (see subsect. 4.3). In this special case a CP–type symmetry may not exist for $\mathcal{L}_H$ since here it requires that both irreps D_Λ and $D_\Lambda \circ \psi_d$ occur in Φ. The more complicated structure of U_H in this case and also for similar cases in $so(8)$ is easily worked out with the methods established in this paper. If $(-D_\Lambda^T) \circ \psi_d \sim D_\Lambda$ and $\psi_d^2 = id$ again the considerations in subsect. 7.1 can be used with $\psi^\triangle$ replaced by ψ_d.

Finally, we want to mention that basis transformations are performed as in the fermionic case eqs. (3.25) and (3.26) but attention has to be paid to two points. In the real case the horizontal part of U_H is real apart from situations discussed in subsect. 7.2. Therefore the corresponding basis transformations have to be performed by orthogonal matrices. Similarly, in the pseudoreal case the transformation matrix must be an element of $Sp(2m)$. These points will be of significance in subsect. 8.3.

8 Yukawa couplings and CP–type symmetries

8.1 The condition on Yukawa couplings

With the vector of scalar field Φ as defined in eq. (7.35) and all fermionic degrees of freedom in a right–handed vector ω_R the most general form of Yukawa couplings is given by

$$\mathcal{L}_Y = \frac{1}{2} i \omega_R^T C^{-1} \Gamma_j \omega_R \Phi_j + h.c. \tag{8.1}$$

where Fermi statistics implies $\Gamma_j^T = \Gamma_j \ \forall\, j$. A CP–type transformation characterized by (R, U, U_H) leads to

$$\text{Condition C:} \qquad U^T \Gamma_k U U^H_{kj} = \Gamma_j^* \qquad \forall\, j \tag{8.2}$$

if we require invariance of $\mathcal{L}_Y$. As a matter of convention we have pulled out the factor $\frac{1}{2}i$ from the coupling matrices Γ_j. The i gives plus signs on both sides of eq. (8.2) whereas the factor 1/2 is motivated by the fact that $\chi_R^T C^{-1} \Gamma_j \chi'_R = \chi'^T_R C^{-1} \Gamma_j \chi_R$, i.e. terms with different multiplets χ_R, χ'_R occurring in ω_R appear twice in $\mathcal{L}_Y$, eq. (8.1).

Before we continue the discussion of Condition C we want to make a few general remarks on CP–type symmetries which we define as those CP–type transformations which fulfill simultaneously Conditions A, B, $\mathrm{B_H}$ and C. First we have the following statements:

$$(R, U, U_H) \text{ CP–type symmetry} \Longleftrightarrow (R^T, U^T, U_H^T) \text{ CP–type symmetry.} \tag{8.3}$$

If (R_i, U_i, U_i^H) $(i = 1, 2)$ are CP–type symmetries then

$$\begin{array}{rcl} \omega_R(x) & \to & -U_1 U_2^* \omega_R(x) \\ \Phi(x) & \to & U_1^H U_2^{H*} \Phi(x) \\ W_\mu^a(x) & \to & (R_1 R_2)_{ab} W_\mu^b(x) \end{array} \tag{8.4}$$

is a symmetry of $\mathcal{L}_G + \mathcal{L}_F + \mathcal{L}_H + \mathcal{L}_Y$. We will see in sect. 9 that eq. (8.4) is a C–type symmetry. As mentioned in the introduction we will not discuss the Higgs potential the inclusion of which would give conditions additional to Conditions A – C. It is clear that theorem II also applies to Condition C.

The coupling matrices Γ_j have two types of indices since they have to tie together irreps and their multiplicities. Therefore, considering matrices which couple irreps D_r,

$D_{r'}$ and D_{r_ϕ} with multiplicities m_r, $m_{r'}$ and m_{r_ϕ}, respectively, where D_r, $D_{r'}$ occur in ω_R and D_{r_ϕ} in Φ we can write Γ_j as a tensor product

$$\Gamma_j = ((\Gamma^a \otimes \gamma^k)^{r_\phi}_{rr'}) \tag{8.5}$$

with j corresponding to the triple (a, k, r_ϕ). Note that $a = 1, \ldots, m_{r_\phi}$, $k = 1, \ldots, d_{r_\phi}$, $(\gamma^k)^{r_\phi}_{rr'}$ are $d_r \times d_{r'}$ matrices and $(\Gamma^a)^{r_\phi}_{rr'}$ $m_r \times m_{r'}$ matrices. Of course, for pseudoreal scalars there are separate coupling matrices for χ and $\widetilde{\chi}$. Clearly, the individual factors of the coupling matrices (8.5) do not have to be symmetric, only the total matrix has to be. For $r = r'$, the diagonal of $(\Gamma^a)^{r_\phi}_{rr}$ couples the same fermion fields to each other and therefore the irrep D_{r_ϕ} has to be contained in the symmetric tensor product $(D_r \otimes D_r)_{\mathrm{sym}}$.

In the further discussion we will confine ourselves to generalized CP invariance and use the following approach which we think is most appropriate in the context of this work. We know already from sect. 5 that CP can be defined in a canonical way for each irrep in ω_R and Φ, respectively, and that a general CP symmetry is composed of the canonical CP transformation $(\mathrm{CP})_0$ eq. (5.5) followed by a horizontal unitary rotation (5.8). Therefore we adopt the strategy that out of the set of CP symmetries of $\mathcal{L}_G + \mathcal{L}_F + \mathcal{L}_H$ we have given a particular one which then imposes Condition C on the Yukawa couplings, i.e., in our strategy the symmetry is primary and determines the Lagrangian.

We will see in subsect. 8.2 that in the CP basis the Clebsch–Gordan coefficients are real. Therefore, $(\gamma^k)^{r_\phi}_{rr'}$, the group–theoretical part of Γ_j, couples the fields in a $(\mathrm{CP})_0$–invariant way and it remains a purely horizontal condition which will be solved in general in subsect. 8.3.

8.2 Real Clebsch–Gordan coefficients and the generalized CP condition

It is easy to see that in a tensor product of two irreps D, D' the Clebsch–Gordan coefficients can be chosen real if we take the tensor product of the respective CP bases. One only has to remember the general procedure for deducing the Clebsch–Gordan series. We additionally imagine that we are in a CP basis where all $D(H_j)$ are diagonal. This is possible because the representation matrices of the CSA can simultaneously be diagonalized by an orthogonal matrix ($D(H_j)$ is symmetric and real) without disturbing symmetry or antisymmetry of the $D(X_a)$ ($a = 1, \ldots, n_G$). Therefore the basis vectors of $D \otimes D'$ given by $\{e_i \otimes e'_j | \, i = 1, \ldots, d; j = 1, \ldots, d'\}$ are all eigenvectors of $D(H_j) \otimes \mathbf{1}_{d'} + \mathbf{1}_d \otimes D'(H_j)$ (e_i, e'_j are the canonical basis vectors of $\mathbf{C}^d$, $\mathbf{C}^{d'}$, respectively). We can e.g. choose $e_1 \otimes e'_1$ to be the unique vector with highest weight $\Lambda + \Lambda'$ in $D \otimes D'$, if Λ, Λ' are the highest weights of D, D', respectively. Applying all $D(e_{-\alpha})$, $\alpha \in \Delta$ or, equivalently, all $D(e_{-\alpha_j})$ ($j = 1, \ldots, \ell$) with α_j simple to $e_1 \otimes e'_1$ gives the representation space associated with $\Lambda + \Lambda'$. Because of eq. (5.13) the basis of this space is real and therefore also the basis of its orthogonal complement can be chosen real. The highest weight in the orthogonal complement can have multiplicity larger than one. We choose a (real) basis vector and make the same procedure as before. We continue along these lines until we have exhausted the whole space $\mathbf{C}^d \otimes \mathbf{C}^{d'}$ [16]. Since all $D(e_{-\alpha})$ are real and we started with a

real basis we finally arrived at a real basis in the Clebsch–Gordan series. Therefore the Clebsch–Gordan coefficients associated with this basis are real.

For the rest of this section we assume that we consider fixed irreps D_r, $D_{r'}$, D_{r_ϕ} with field multiplets χ_R, χ'_R, ϕ, respectively. Then one can readily check that

$$(CP)_0: \ \chi_R \to -C\chi_R^*, \ \chi'_R \to -C\chi'^*_R, \ \phi \to \phi^* \tag{8.6}$$

is a symmetry of

$$i\chi_R^T C^{-1}\gamma^k \chi'_R \phi_k + h.c. \tag{8.7}$$

for real Clebsch–Gordan matrices γ^k since the invariance condition is just

$$i\chi_R^T C^{-1}\gamma^k \chi'_R \phi_k \stackrel{(CP)_0}{\longrightarrow} -i\chi'^\dagger_R C\gamma^{kT}\chi_R^*\phi_k^* = (i\chi_R^T C^{-1}\gamma^k\chi'_R\phi_k)^\dagger \tag{8.8}$$

or $\gamma^{kT} = \gamma^{k\dagger}$. Consequently, taking into account the multiplicities m_r, $m_{r'}$, m_{r_ϕ} of the respective irreps a generalized CP transformation given by

$$CP: \ \chi_R \to -UC\chi_R^*, \ \chi'_R \to -U'C\chi'^*_R, \ \phi \to H\phi^* \tag{8.9}$$

leads to the condition

$$U^T\Gamma^b U' H_{ba} = \Gamma^{a*} \tag{8.10}$$

in the horizontal spaces on which U, U', H act. Since CP does not connect different irreps the above discussion is also fully general. Of course, the canonical CP symmetry $U = \mathbf{1}$, $U' = \mathbf{1}'$, $H = \mathbf{1}_\phi$ would simply require real matrices Γ^a. Thus we have achieved to separate group indices and horizontal indices in the case of generalized CP invariance. The classes of solutions of eq. (8.10) will be discussed in the following subsection.

8.3 Solutions of the generalized CP condition

The discussion of eq. (8.10) is greatly simplified by the freedom of choosing suitable bases in the horizontal spaces. As in the previous subsection we stick to fixed irreps D_r, $D_{r'}$, D_{r_ϕ} and therefore we have three independent basis transformations (3.26) represented by the unitary matrices Z, Z', Z_ϕ. According to the theorem proved in ref. [29] one can find Z and Z' such that

$$\begin{aligned} Z^\dagger U Z^* &= \mathrm{diag}\,(O(\Theta_1),\ldots,O(\Theta_k),\mathbf{1}_p), \qquad 0 < \Theta_\nu \leq \frac{\pi}{2} \\ Z'^\dagger U' Z'^* &= \mathrm{diag}\,(O(\Theta'_1),\ldots,O(\Theta'_{k'}),\mathbf{1}_{p'}), \qquad 0 < \Theta'_\nu \leq \frac{\pi}{2} \end{aligned} \tag{8.11}$$

with

$$O(\vartheta) \equiv \left(\begin{array}{cc} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{array} \right). \tag{8.12}$$

In the Higgs sector we have to distinguish the cases as in sect. 7. For a complex scalar multiplet the above theorem applies yielding

$$Z_\phi^\dagger H Z_\phi^* = \mathrm{diag}\,(O(\Theta_1^H),\ldots,O(\Theta_{k_H}^H),\mathbf{1}_{p_H}), \qquad 0 < \Theta_\nu^H \leq \frac{\pi}{2}. \tag{8.13}$$

For a real scalar H is real and Z_ϕ has to be orthogonal. Then the real version of the spectral theorem for normal operators tells that one can achieve

$$Z_\phi^T H Z_\phi = \text{diag}\ (O(\Theta_1^H), \ldots, O(\Theta_{k_H}^H), -\mathbf{1}_{p_-}, \mathbf{1}_{p_+}), \qquad 0 < \Theta_\nu^H < \pi. \tag{8.14}$$

The case of a pseudoreal scalar requires $Z_\phi \in Sp(2m)$. In app. G we prove that for any $H \in Sp(2m)$ there is such a Z_ϕ giving

$$Z_\phi^\dagger H Z_\phi^* = \left(\begin{array}{cc|cc} 0 & 0 & D & 0 \\ 0 & \mathbf{1}_{p_H} & 0 & 0 \\ \hline -D^* & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{p_H} \end{array}\right) \tag{8.15}$$

with $D = \text{diag}\ (d_1, \ldots, d_{k_H})$, $|d_\nu| = 1$.

These basis choices allow to solve eq. (8.10) in a piecewise manner with submatrices of Γ^a of maximal size 2×2 and at most two different Γ^a involved at a time. We will denote these submatrices by A or A_1, A_2 and indicate by arrows which part of $Z^\dagger U Z^*$, $Z'^\dagger U' Z'^*$, $Z_\phi^\dagger H Z_\phi^*$ is under discussion.

The number of different cases to be discussed is reduced by the following observations. Solutions of the cases with $H \to -1$ are obtained from those with $H \to 1$ by multiplying A or A_1, A_2 by i. The solutions of the cases $U \to 1$, $U' \to O(\Theta')$ are obtained from those with $U \to O(\Theta)$, $U' \to 1$ by transposition of A or A_1, A_2. This leaves the following nine generic cases to be investigated:

1) $U \to 1$, $U' \to 1 \Rightarrow A, A_1, A_2$ are 1×1 matrices

1a) $H \to 1$: $A = A^*$

1b) $H \to O(\Theta_H)$: $\begin{array}{l} A_1 \cos\Theta_H - A_2 \sin\Theta_H = A_1^* \\ A_1 \sin\Theta_H + A_2 \cos\Theta_H = A_2^* \end{array}$

1c) $H \to h(d)$: $\begin{array}{r} A_1 d = A_2^* \\ -A_2 d^* = A_1^* \end{array}$

2) $U \to O(\Theta)$, $U' \to 1 \Rightarrow A, A_1, A_2$ are 2×1 matrices

2a) $H \to 1$: $O(\Theta)^T A = A^*$

2b) $H \to O(\Theta_H)$: $\begin{array}{l} O(\Theta)^T (A_1 \cos\Theta_H - A_2 \sin\Theta_H) = A_1^* \\ O(\Theta)^T (A_1 \sin\Theta_H + A_2 \cos\Theta_H) = A_2^* \end{array}$

2c) $H \to h(d)$: $\begin{array}{r} O(\Theta)^T A_1 d = A_2^* \\ -O(\Theta)^T A_2 d^* = A_1^* \end{array}$

3) $U \to O(\Theta)$, $U' \to O(\Theta') \Rightarrow A, A_1, A_2$ are 2×2 matrices

3a) $H \to 1$: $O(\Theta)^T A O(\Theta') = A^*$

$$\text{3b)}\ \ H \to O(\Theta_H): \begin{array}{l} O(\Theta)^T(A_1\cos\Theta_H - A_2\sin\Theta_H)O(\Theta') = A_1^* \\ O(\Theta)^T(A_1\sin\Theta_H + A_2\cos\Theta_H)O(\Theta') = A_2^* \end{array}$$

$$\text{3c)}\ \ H \to h(d): \begin{array}{r} O(\Theta)^T A_1 O(\Theta')d = A_2^* \\ -O(\Theta)^T A_2 O(\Theta')d^* = A_1^*. \end{array} \tag{8.16}$$

In all the cases we have

$$0 < \Theta,\ \Theta' \leq \frac{\pi}{2}, \qquad 0 < \Theta_H < \pi, \qquad h(d) = \begin{pmatrix} 0 & d \\ -d^* & 0 \end{pmatrix} \text{ with } |d| = 1.$$

In the following the solutions for A or A_1, A_2 of 1a) – 3c) are given as functions of Θ, Θ', Θ_H or d according to the spirit of our strategy outlined at the end of subsect. 8.1. The methods used thereby can be found in app. H. Some cases in the above list correspond to each other like 1b) – 2a) and 2b) – 3a). They are, however, kept apart for the sake of clearness. In the following list of solutions ε denotes the two options ± 1.

Solutions:

1a) $A \in \mathbf{R}$

1b) $A_1 = A_2 = 0$

1c) $d = i\varepsilon \Rightarrow A_2 = -dA_1^*,\ A_1 \in \mathbf{C}$

$d^2 \neq -1 \Rightarrow A_1 = A_2 = 0$

2a) $A = 0$

2b) $\Theta_H = \Theta = \dfrac{\pi}{2} \Rightarrow A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A_1^*,\ A_1 \in \mathbf{C}^2$

$\Theta_H = \Theta \neq \dfrac{\pi}{2} \Rightarrow A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A_1,\ A_1 \in \mathbf{R}^2$

$\Theta_H = \pi - \Theta \neq \dfrac{\pi}{2} \Rightarrow A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A_1,\ iA_1 \in \mathbf{R}^2$

$\Theta_H \notin \{\Theta,\ \pi - \Theta\} \Rightarrow A_1 = A_2 = 0$

2c) $\Theta = \dfrac{\pi}{2},\ d = \varepsilon \Rightarrow A_2 = \varepsilon \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A_1^*,\ A_1 \in \mathbf{C}^2$

$\Theta \neq \dfrac{\pi}{2},\ d = i\varepsilon e^{\pm i\Theta} \Rightarrow A_1 = a \begin{pmatrix} 1 \\ \pm i \end{pmatrix},\ A_2 = -i\varepsilon A_1^*,\ a \in \mathbf{C}$

$d^2 \neq -e^{\pm 2i\Theta} \Rightarrow A_1 = A_2 = 0$

3a) $\Theta=\Theta'=\dfrac{\pi}{2}\Rightarrow A=\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix},\ a,b\in\mathbf{C}$

$\Theta=\Theta'\neq\dfrac{\pi}{2}\Rightarrow A=\begin{pmatrix} a & b \\ -b & a \end{pmatrix},\ a,b\in\mathbf{R}$

$\Theta\neq\Theta'\Rightarrow A=0$

3b) $\Theta_H=\Theta+\Theta'=\dfrac{\pi}{2}\Rightarrow A_1=\begin{pmatrix} a & b \\ b & -a \end{pmatrix},\ A_2=\begin{pmatrix} -b & a \\ a & b \end{pmatrix}^*,\ a,b\in\mathbf{C}$

$\cos\Theta_H=\varepsilon\sin\Theta',\ \Theta=\dfrac{\pi}{2},\ \Theta'\neq\dfrac{\pi}{2}\Rightarrow A_1=\begin{pmatrix} a & b \\ -\varepsilon b^* & \varepsilon a^* \end{pmatrix},\ A_2=\begin{pmatrix} \varepsilon b & -\varepsilon a \\ a^* & b^* \end{pmatrix}$
$a,b\in\mathbf{C}$

$\cos\Theta_H=\varepsilon\sin\Theta,\ \Theta'=\dfrac{\pi}{2},\ \Theta\neq\dfrac{\pi}{2}\Rightarrow A_1=\begin{pmatrix} a & b \\ -\varepsilon b^* & \varepsilon a^* \end{pmatrix},\ A_2=\begin{pmatrix} -b^* & a^* \\ -\varepsilon a & -\varepsilon b \end{pmatrix}$
$a,b\in\mathbf{C}$

$\cos\Theta_H=-\varepsilon\cos(\Theta+\Theta'),\ \Theta\neq\dfrac{\pi}{2},\ \Theta'\neq\dfrac{\pi}{2},\ \Theta+\Theta'\neq\dfrac{\pi}{2}\Rightarrow$

$$\left.\begin{array}{lll} A_1=i\begin{pmatrix} a & b \\ b & -a \end{pmatrix}, & A_2=-i\begin{pmatrix} -b & a \\ a & b \end{pmatrix} & \text{for } \varepsilon=1 \\ A_1=\begin{pmatrix} a & b \\ b & -a \end{pmatrix}, & A_2=\begin{pmatrix} -b & a \\ a & b \end{pmatrix} & \text{for } \varepsilon=-1 \end{array}\right\} a,b\in\mathbf{R}$$

$\cos\Theta_H=\varepsilon\cos(\Theta-\Theta'),\ \Theta\neq\dfrac{\pi}{2},\ \Theta'\neq\dfrac{\pi}{2},\ \Theta+\Theta'\neq\dfrac{\pi}{2}\Rightarrow$

$$\left.\begin{array}{lll} A_1=\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, & A_2=\eta\begin{pmatrix} -b & a \\ -a & -b \end{pmatrix} & \text{for } \varepsilon=1 \\ A_1=i\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, & A_2=-i\eta\begin{pmatrix} -b & a \\ -a & -b \end{pmatrix} & \text{for } \varepsilon=-1 \end{array}\right\} \begin{array}{l} a,b\in\mathbf{R}\text{ and} \\ \eta=\operatorname{sgn}(\Theta'-\Theta) \end{array}$$

All other choices of Θ, Θ', Θ_H lead to $A_1=A_2=0$.

3c) $\Theta=\Theta'=\dfrac{\pi}{2},\ d=i\varepsilon\Rightarrow A_1=\begin{pmatrix} a & b \\ c & d \end{pmatrix},\ A_2=-i\varepsilon\begin{pmatrix} d^* & -c^* \\ -b^* & a^* \end{pmatrix},\ a,b,c,d\in\mathbf{C}$

$\Theta=\Theta'\neq\dfrac{\pi}{2},\ d=i\varepsilon\Rightarrow A_1=\begin{pmatrix} a & b \\ -b & a \end{pmatrix},\ A_2=-i\varepsilon\begin{pmatrix} a^* & b^* \\ -b^* & a^* \end{pmatrix},\ a,b\in\mathbf{C}$

$\Theta+\Theta'=\dfrac{\pi}{2},\ d=\varepsilon\Rightarrow A_1=\begin{pmatrix} a & b \\ b & -a \end{pmatrix},\ A_2=\varepsilon\begin{pmatrix} -b^* & a^* \\ a^* & b^* \end{pmatrix},\ a,b\in\mathbf{C}$

$\Theta'=\dfrac{\pi}{2},\ \Theta\neq\dfrac{\pi}{2},\ d=\varepsilon e^{-i\Theta}\Rightarrow A_1=\begin{pmatrix} a & ib \\ -ia & b \end{pmatrix},\ A_2=\varepsilon\begin{pmatrix} ib^* & a^* \\ -b^* & ia^* \end{pmatrix},\ a,b\in\mathbf{C}$

$\Theta=\dfrac{\pi}{2},\ \Theta'\neq\dfrac{\pi}{2},\ d=\varepsilon e^{-i\Theta'}\Rightarrow A_1=\begin{pmatrix} a & -ia \\ ib & b \end{pmatrix},\ A_2=\varepsilon\begin{pmatrix} ib^* & -b^* \\ a^* & ia^* \end{pmatrix},\ a,b\in\mathbf{C}$

$$\Theta = \frac{\pi}{2},\ \Theta' \neq \frac{\pi}{2},\ d = \varepsilon e^{i\Theta'} \Rightarrow A_1 = \begin{pmatrix} a & ia \\ -ib & b \end{pmatrix},\ A_2 = \varepsilon \begin{pmatrix} -ib^* & -b^* \\ a^* & -ia^* \end{pmatrix},\ a, b \in \mathbf{C}$$

$$\Theta' = \frac{\pi}{2},\ \Theta \neq \frac{\pi}{2},\ d = \varepsilon e^{i\Theta} \Rightarrow A_1 = \begin{pmatrix} a & -ib \\ ia & b \end{pmatrix},\ A_2 = \varepsilon \begin{pmatrix} -ib^* & a^* \\ -b^* & -ia^* \end{pmatrix},\ a, b \in \mathbf{C}$$

The remaining four non–trivial cases can be uniformly described in the following way:

$$\Theta \neq \frac{\pi}{2}, \qquad \Theta' \neq \frac{\pi}{2}, \qquad d = i\varepsilon e^{-i(r\Theta + s\Theta')} \Rightarrow A_1 = c v_r v_s^T, \qquad A_2 = -i\varepsilon A_1^*$$

with $r, s = \pm$, $v_\pm = \begin{pmatrix} 1 \\ \mp i \end{pmatrix}$, $c \in \mathbf{C}$.

For all other choices of Θ, Θ' and d we have $A_1 = A_2 = 0$.

This concludes the complete discussion of the generalized CP condition (8.10). It shows which solutions apart from the trivial one with real couplings 1a) one can expect. These solutions might be helpful for model building. Of course, they are bound to the bases introduced in the beginning of this subsection. If one has additional conditions on the Yukawa couplings from further symmetries it might not be useful to work in these bases. An example for such a case is given in subsect. 9.3.

9 C–type transformations

9.1 Charge conjugation

We have seen in eq. (8.4) that if we carry out two CP–type transformations one after the other we obtain a transformation of the type [7]

$$\begin{array}{rcl} W_\mu^a(x) & \to & R_{ab} W_\mu^b(x) \\ \omega_R(x) & \to & U \omega_R(x) \\ \Phi(x) & \to & U_H \Phi(x). \end{array} \tag{9.1}$$

It is reasonable to call eq. (9.1) a C–type transformation since composing CP with P, both of CP–type, should give charge conjugation. Though the discussion of CP and P in the previous sections contains implicitly also C–type transformations it is nevertheless useful to consider their features in a separate section.

If $\mathcal{L}_G + \mathcal{L}_F + \mathcal{L}_H + \mathcal{L}_Y$ is invariant under the transformation (9.1) we have

$$\begin{array}{rcl} U T_b R_{ba} U^\dagger & = & T_a \\ U_H \mathcal{T}_b R_{ba} U_H^\dagger & = & \mathcal{T}_a \\ U^T \Gamma_k U U_{kj}^H & = & \Gamma_j. \end{array} \tag{9.2}$$

As we have learned in subsect. 3.2 the first relation can be interpreted as $\{T_a\}$ composed with the automorphism ψ_R being equivalent to $\{T_a\}$. The second relation has the analogous interpretation for the Higgs fields. It is clear from Schur's lemma that for $R = \mathbf{1}$ the matrices U and U_H only act horizontally.

To define a charge conjugation we require as for P and CP that $\psi_R = \psi_C$ is a non–trivial involution. Having fixed the CSA $\mathcal{H}$ which determines the quantum numbers of the states in the irreps it is reasonable to require $\psi_C(\mathcal{H}) = \mathcal{H}$. As in the case of P the CSA can be split into $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ with $\psi_C(X) = \pm X$ for $X \in \mathcal{H}_\pm$. We assume that the CSA of the unbroken part of the SM group is contained in $\mathcal{H}_-$, i.e. Q_{em}, F_3^c, $Y_c \in \mathcal{H}_-$ as any physically viable model must have $U(1)_{em} \times SU(3)_c \subseteq G$. Thus ψ_C has to flip the sign of at least three generators in $\mathcal{H}$. To every ψ_C there is associated a subgroup of G generated by the elements of $\mathcal{L}_C$ with $\psi_C(X) = X$. In refs. [7, 35] there is a complete list of all such "symmetric subgroups" S for simple groups G. Since in this paper we are mainly concerned with the symmetry aspect of the Lagrangians and not with the embedding of the SM group in G and future symmetry breaking we refer the reader to the extensive discussion in ref. [7] of these questions.

Let the automorphism defining charge conjugation be given by

$$\psi_C(H_j) = \rho_j H_j = \begin{cases} -H_j, & j = 1, \ldots, p \\ H_j, & j = p+1, \ldots, \ell. \end{cases} \tag{9.3}$$

Then as for P we can distinguish an internal and external case with respect to an irrep D if $D \sim D \circ \psi_C$ or $D \not\sim D \circ \psi_C$, respectively. As before we consider now the representation space of D in the internal case and the direct sum of the spaces of D and $D \circ \psi_C$ in the external case. Then

$$D(H_j) U_C e(\lambda, q) = \rho_j \lambda(H_j) U_C e(\lambda, q) \tag{9.4}$$

for the ON basis $\{e(\lambda, q)\}$ with weights λ of D. The situation is analogous to P in eqs. (6.8) and (6.10). All states with $(\lambda(H_1), \ldots, \lambda(H_p)) \neq (0, \ldots, 0)$ have a counterpart with opposite quantum numbers with respect to $H_1, \ldots, H_p$. Defining

$$\omega_L^{\lambda,q}(x) \equiv C\gamma_0^T (\omega_R^{\lambda_C,q}(x))^* \tag{9.5}$$

with λ_C given as in eq. (6.10) we can perform an analysis as in subsect. 6.2 and obtain

$$\mathcal{U}_C \omega_R^{\lambda,q}(x) \mathcal{U}_C^{-1} = e^{i\delta} C\gamma_0^T (\omega_L^{\lambda,q}(x))^*. \tag{9.6}$$

Here we have used that U_C^2 is a phase in each irrep of $\{T_a\}$ just as in the analogous situation in eq. (6.15)[3]. In eq. (9.6) charge conjugation has the familiar form.

As mentioned before there is a close relationship between C and P via CP. It is clear that invariance under CP and P is equivalent to invariance under CP and C=CP∘P. This relationship is given by the identifications $\psi_C = \psi^\triangle \circ \psi_P$, $\lambda_P = \lambda_C$ and $U_C = -U_P U_{CP}^*$ (see eq. (8.4)). Once we fix $\psi_{CP} = \psi^\triangle$ and choose a CP basis one can therefore identify U_P with U_C in the internal case of parity (see, e.g., the $SO(10)$ example in sect.2).

[3] δ is the phase of U_C^2 in D or $D \circ \psi_C$ for the internal or external case, respectively.

Like parity, charge conjugation cannot be defined in the SM where ψ_C would either reverse the sign of the electric charge and colour charges or of all the four elements of the CSA. In any case the right–handed singlets have no partners with opposite charges as required by the presence of $D \circ \psi_C$ in the case of C invariance.

In the light of the discussion in this section we also learn that in the irreps of G^*, eq. (4.10), $D(E)$ would be a candidate for U_C if $\psi_C = \psi_E$.

9.2 Compatibility of CP and C

In this subsection we will show that one can construct Yukawa couplings which are not only invariant under the canonical CP transformation but also under a "canonical" charge conjugation at the same time. We will make use of the advantages of the CP basis where $(\mathrm{CP})_0$ is simply given by eq. (8.6) and where the Yukawa couplings are real.

To define a canonical charge conjugation we confine ourselves to D and $D \circ \psi_C$ for every irrep D contained in $\{T_a\}$. Given the involutive automorphism ψ_C we will assume in the following that either $D \circ \psi_C \sim D$ or $D \circ \psi_C$ is included in $\{T_a\}$ to allow for a definition of C. The whole discussion will be confined to simple Lie algebras.

It is clear that for $\psi_C \equiv \psi_Y$ inner (see eq. (4.1)) the C transformation

$$\chi_R \to W\chi_R, \qquad \chi'_R \to W'\chi'_R, \qquad \phi \to W_\phi \phi \tag{9.7}$$

with

$$W = e^{-D(Y)}, \qquad W' = e^{-D'(Y)}, \qquad W_\phi = e^{-D_\phi(Y)} \tag{9.8}$$

where the irreps D, D', D_ϕ are coupled together in $\mathcal{L}_Y$ is just a gauge transformation and $\mathcal{L}_Y$ is obviously invariant under transformation (9.7). This allows to restrict the further discussion to $\psi = \psi^\triangle$ or ψ_d. In the following the matrices acting on the minimal number of irreps involved in the Yukawa couplings for a definition of C will always be called W, W', W_ϕ. They are part of U_C and U_C^H in eq. (9.1).

In a CP basis $-D^T = D \circ \psi^\triangle$ is valid and therefore, if $D \sim D \circ \psi^\triangle$, we obtain W real, $W^T = \lambda W$ with $\lambda = -1$ for D pseudoreal and $\lambda = 1$ for D real (see eqs. (7.12) and (7.19), respectively). For W', W_ϕ the parameters analogous to λ will be denoted by λ', λ_ϕ, respectively. If $D \sim D \circ \psi_d$ it is shown in app. I that W is real and symmetric, i.e. $\lambda = 1$. As discussed in sect. 4 we choose ψ_d as the relevant outer automorphism for $so(2\ell)$ ($\ell = 4, 6, 8, \ldots$) whereas in all other cases of non–trivial outer involutive automorphisms, i.e., the Lie algebras $su(\ell+1)$ ($\ell \geq 2$), $so(2\ell)$ ($\ell = 5, 7, 9, \ldots$) and cE_6, we take $\psi^\triangle$ as the relevant automorphism.

We have to distinguish four cases according to the types of irreps with respect to ψ_C coupled together in $\mathcal{L}_Y$:

a) $D \sim D \circ \psi_C$, $D' \sim D' \circ \psi_C$, $D_\phi \sim D_\phi \circ \psi_C$

The $\mathcal{L}_Y$ of eq. (8.7) transforms as

$$i\chi_R^T C^{-1} \gamma^k \chi'_R \phi_k \to i\chi_R^T C^{-1} W^T \gamma^\ell W' \chi'_R W_{\phi\ell k} \phi_k \tag{9.9}$$

under the C transformation (9.7). In case a) we always have

$$\lambda\lambda'\lambda_\phi = 1. \tag{9.10}$$

This is so for two reasons. First, for ψ_d the W's are symmetric and all λ's are 1. Second, as for $\psi^\triangle$ one can show that in a tensor product of real or pseudoreal irreps of connected semisimple Lie groups pseudoreal irreps do not occur. Also if one factor is real and the other one pseudoreal then real irreps do not occur [34] and thus eq. (9.10) is valid. Consequently, we can replace γ^k by

$$\gamma^k \to \gamma^k_\pm \equiv \gamma^k \pm W\gamma^\ell W'^T (W_\phi^T)_{\ell k} \quad \forall k \tag{9.11}$$

with the C transformation given by

$$aW,\ a'W',\ a_\phi W_\phi \qquad \text{with} \qquad aa'a_\phi = \begin{cases} 1 & \text{for } + \\ -1 & \text{for } - \,. \end{cases} \tag{9.12}$$

Clearly, eq. (9.10) is necessary to have invariance of $\mathcal{L}_Y$ eq. (8.7) with $\gamma^k_\pm$ given by eq. (9.11). Since the coupling matrices (9.11) are real $(\mathrm{CP})_0$ invariance is not spoiled. We do not know if one can take a specific sign for all simple groups and all irreps. In any case, in any situation the two sets of coupling matrices $\{\gamma^k_+\}$ and $\{\gamma^k_-\}$ cannot consist of only zero matrices at the same time and therefore it is possible to have $(\mathrm{CP})_0$ invariance *and* invariance under the "canonical" C transformation given by eq. (9.12) for $\mathcal{L}_Y$ redefined with $\{\gamma^k_+\}$ or $\{\gamma^k_-\}$ in case a).

b) $D \not\sim D\circ\psi_C$, $D' \not\sim D'\circ\psi_C$, $D_\phi \not\sim D_\phi\circ\psi_C$

In this case

$$W, W', W_\phi = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad \chi_R = \begin{pmatrix} \chi_{R1} \\ \chi_{R2} \end{pmatrix} \text{ etc.} \tag{9.13}$$

Then

$$i(\chi^T_{R1} C^{-1}\gamma^k \chi'_{R1}\phi^1_k + \chi^T_{R2} C^{-1}\gamma^k \chi'_{R2}\phi^2_k) \tag{9.14}$$

is clearly invariant under $(\mathrm{CP})_0$ and C given by eq. (9.13).

c) $D \sim D\circ\psi_C$, $D' \sim D'\circ\psi_C$, $D_\phi \not\sim D_\phi\circ\psi_C$

With

$$i(\chi^T_R C^{-1}\gamma^k \chi'_R\phi^1_k + \chi^T_R C^{-1} W\gamma^k W'^T \chi'_R\phi^2_k) \tag{9.15}$$

and

$$W_\phi = \begin{pmatrix} 0 & \lambda\lambda'\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad \phi = \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix} \tag{9.16}$$

invariance is obtained.

c') $D' \sim D' \circ \psi_C$, $D_\phi \sim D_\phi \circ \psi_C$, $D \not\sim D \circ \psi_C$

As before we have

$$i(\chi_{R1}^T C^{-1} \gamma^k \chi'_R \phi_k + \chi_{R2}^T C^{-1} \gamma^\ell W'^T \chi'_R (W_\phi^T)_{\ell k} \phi_k) \tag{9.17}$$

with

$$W = \begin{pmatrix} 0 & \lambda' \lambda_\phi \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad \chi_R = \begin{pmatrix} \chi_{R1} \\ \chi_{R2} \end{pmatrix}. \tag{9.18}$$

d) $D \not\sim D \circ \psi_C$, $D' \not\sim D' \circ \psi_C$, $D_\phi \sim D_\phi \circ \psi_C$

Now

$$i(\chi_{R1}^T C^{-1} \gamma^k \chi'_{R1} + \chi_{R2}^T C^{-1} \gamma^\ell (W_\phi^T)_{\ell k} \chi'_{R2}) \phi_k \tag{9.19}$$

is invariant with $(a, a' = \pm 1)$

$$W = \begin{pmatrix} 0 & a\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad W' = \begin{pmatrix} 0 & a'\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad \lambda_\phi = aa'. \tag{9.20}$$

All remaining cases are to be discussed analogously. Thus we have seen that for every C transformation such that $D \circ \psi_C$ is contained in $\{T_a\}$ for every irrep D of $\{T_a\}$ one can write down Yukawa couplings invariant under canonical C and CP with "canonical C" defined in this subsection. This invariance is achieved by a redefinition of the γ^k. It is not clear to us whether one could circumvent this redefinition by directly exploring properties of the Clebsch–Gordan coefficients.

9.3 An example with $G = SU(2)_L \times SU(2)_R \times U(1)_{B-L}$

We want to close this section with a left–right symmetric example [13, 32, 33] which exhibits a lot of interesting features in the light of the discussion of discrete symmetries. We consider fermions χ_L, χ_R transforming as $(2, 1, B - L)$, $(1, 2, B - L)$, respectively, where the numbers indicate the dimension $2j + 1$ of an irrep D_j $(j = 0, 1/2, 1, \ldots)$ of $SU(2)$. The scalar ϕ_m transforming as (2,2,0) was already introduced in subsect. 7.2 as an example of two real irreps in complex disguise. For quarks the $U(1)$ charge is 1/3 whereas for leptons it is -1. Therefore it is given by baryon number minus lepton number $(B-L)$. Assuming n_f families in $\chi_{L,R}$ we have

$$\omega_R = \begin{pmatrix} (\chi_L)^c \\ \chi_R \end{pmatrix} \tag{9.21}$$

and

$$-\mathcal{L}_Y = \bar{\chi}_L \Gamma \phi_m \chi_R + \bar{\chi}_L \Delta \widetilde{\phi}_m \chi_R + h.c. \tag{9.22}$$

with $\widetilde{\phi}_m$ given by eq. (7.28). Γ and Δ are $n_f \times n_f$ matrices in family (flavour) space. Note that in eq. (9.22) both ϕ_m and $\widetilde{\phi}_m$ have to be coupled to get the most general Yukawa couplings.

Let us now investigate the effects of the following generalized CP and C transformations [27]:

$$CP \;:\; \chi_L \to -C\chi_L^*, \quad \chi_R \to -iC\chi_R^*, \quad \phi_m \to -i\phi_m^* \tag{9.23}$$

$$C \;:\; \chi_L \to -(\chi_R)^c, \quad \chi_R \to -i(\chi_L)^c, \quad \phi_m \to -i\phi_m^T. \tag{9.24}$$

Examining the invariance conditions we obtain

$$CP \;\Rightarrow\; \Gamma \text{ real, } \Delta \text{ imaginary,} \tag{9.25}$$

$$C \;\Rightarrow\; \Gamma^T = \Gamma,\ \Delta^T = -\Delta. \tag{9.26}$$

Taking eqs. (9.25) and (9.26) together we see that Γ has to be real and symmetric and Δ imaginary and antisymmetric. Here we have not used the bases in the horizontal space introduced in subsect. 8.3 for CP because there is the additional C invariance (9.24) whose simple form would be spoiled in these bases. Furthermore, eqs. (9.25) and (9.26) together are rather restrictive. It was shown in ref. [27] that the effect of these conditions is similar to the effect of a horizontal symmetry allowing for relations between flavour mixing and masses but for $n_f = 3$ the top quark mass comes out too low.

Interpreting the transformation properties of fermions and ϕ_m in the light of the irreps of G^* in subsect. 4.2 and ignoring the $U(1)$, ω_R eq. (9.21) is in the irrep $D_{1/2,0}$ of $SU(2)^*$ and ϕ_m contains two $D^+_{1/2}$. Thus ϕ_m can be considered as a tensor product $D_{1/2} \otimes D_{1/2}$ with respect to $SU(2) \times SU(2)$ with $D(E)$ exchanging the vectors in the product. Since $D(E)$ is related to the C in eq. (9.24) it becomes clear that C has the effect of transposition on ϕ_m. Combining CP and C we obtain

$$P = C \circ CP : \chi_L \to -\gamma_0\chi_R, \quad \chi_R \to -\gamma_0\chi_L, \quad \phi_m \to \phi_m^\dagger. \tag{9.27}$$

Now we want to switch to the formalism discussed in this paper, namely using ω_R eq. (9.21) and ϕ eq. (7.29) instead of ϕ_m. Then the fermion rep has the form

$$T_a = \begin{pmatrix} 0 & 0 \\ 0 & \tau_a/2 \end{pmatrix}, \quad T_{3+a} = \begin{pmatrix} -\tau_a^T/2 & 0 \\ 0 & 0 \end{pmatrix}, a = 1,2,3, \quad T_7 = \frac{1}{2}(B-L)\begin{pmatrix} -\mathbf{1}_2 & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix} \tag{9.28}$$

with Pauli matrices τ_a. The generators $\mathcal{T}_a$ acting on ϕ are obtained from the transformation property of ϕ_m eq. (7.27), i.e., $\mathcal{T}_a\phi$ $(a = 1,2,3)$ corresponds to $-\phi_m\tau_a/2$ and $\mathcal{T}_{3+a}\phi$ to $\tau_a\phi_m/2$. Therefore we obtain

$$\begin{aligned}
&\mathcal{T}_1 = -\frac{1}{2}\begin{pmatrix} 0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0 \end{pmatrix}, \qquad \mathcal{T}_2 = -\frac{1}{2}\begin{pmatrix} 0&0&i&0\\0&0&0&i\\-i&0&0&0\\0&-i&0&0 \end{pmatrix}, \\
&\mathcal{T}_3 = -\frac{1}{2}\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1 \end{pmatrix}, \quad \mathcal{T}_{3+a} = \tfrac{1}{2}\begin{pmatrix} \tau_a & 0 \\ 0 & \tau_a \end{pmatrix}, \qquad \mathcal{T}_7 = 0.
\end{aligned} \tag{9.29}$$

Now CP eq. (9.23) is quickly treated. Acting on ω_R, eq. (3.17) determines

$$U_{CP} = \begin{pmatrix} -\mathbf{1}_{n_f} & 0 \\ 0 & i\mathbf{1}_{n_f} \end{pmatrix}. \tag{9.30}$$

Then the automorphism is fixed by Condition B:

$$R_{CP} = \mathrm{diag}\,(-1, 1, -1, -1, 1-1, -1) \tag{9.31}$$

is just $\psi^{\triangle}$ for $SU(2) \times SU(2) \times U(1)$. For ϕ we have $U^H_{CP} = -i\mathbf{1}_4$ and a quick look at eq. (9.29) confirms that $-\mathcal{T}_a^T R_{CPaa} = \mathcal{T}_a$ is fulfilled.

As for charge conjugation eq. (9.24) fixes

$$U_C = -\begin{pmatrix} 0 & \mathbf{1}_{n_f} \\ i\mathbf{1}_{n_f} & 0 \end{pmatrix} \qquad \text{and} \qquad R_C = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{9.32}$$

is then given by eq. (9.2). Now we have to check consistency in the Higgs sector where from eq. (9.24) we calculate

$$U^H_C = -i \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{9.33}$$

It is then tedious but easy to verify that

$$U^H_C(\mathcal{T}_b R_{Cba}) U^{H\dagger}_C = \mathcal{T}_a.$$

This was a somewhat unorthodox look at the discrete symmetries CP, C and P in left–right symmetric models. In this example P has the simple form coming from the idea of left–right symmetry whereas CP and C are "generalized" in the sense of subsect. 5.2 because there is a phase in one part of the horizontal space. The general discussion in this paper is quite nicely illustrated here and though the gauge group is rather small the structure of the discrete symmetries is more complex than in the cases of QED and QCD.

10 Comments and conclusions

In this work we have discussed the possibility of defining CP, P and C invariance in gauge theories before spontaneous symmetry breaking. As mentioned in the introduction the

breaking of the gauge group and of the discrete symmetries could be performed at the same time. For CP this possibility was first envisaged in ref. [36] and for P in refs. [37, 32].

Our first comment concerns the question of when the above scenario really happens. Let us suppose that the Lagrangian is invariant under a CP transformation where U_H is the unitary matrix appearing in the transformation of the scalar field vector Φ eq. (7.37). Then, if the vacuum expectation value $\langle\Phi\rangle_0$ fulfills

$$U_H \langle\Phi\rangle_0^* = \langle\Phi\rangle_0 \tag{10.1}$$

the CP symmetry is not broken [38, 39, 40]. Furthermore, if eq. (10.1) is not fulfilled there are two possibilities. On the one hand, there might exist an element of the total symmetry group $G \times H$ of the Lagrangian (G is the gauge group and H the group of all other internal symmetry transformations commuting with G) such that its action on Φ is given by S_H and

$$S_H U_H \langle\Phi\rangle_0^* = \langle\Phi\rangle_0. \tag{10.2}$$

Then one can define a new CP symmetry

$$\Phi(x) \to S_H U_H \Phi(\hat{x})^* \tag{10.3}$$

with adequate redefinitions in the fermion and gauge boson sectors and eq. (10.1) is fulfilled with U_H replaced by $U'_H = S_H U_H$. On the other hand, only if for a given U_H one cannot find an S_H such that U'_H complies with eq. (10.1) the CP symmetry associated with the U_H of eq. (10.1) is spontaneously broken together with the gauge group.

The second comment refers to CPT invariance. One might be tempted to define a CPT–type transformation in analogy to CP–type transformations defined in sect. 3 by

$$\begin{array}{rcl} W_\mu^a(x) & \to & -R_{ab} W_\mu^b(-x) \\ \omega_R(x) & \to & -U^T \gamma_5^T \omega_R(-x)^* \\ \Phi(x) & \to & -U_H^T \Phi(-x)^*. \end{array} \tag{10.4}$$

It can easily be checked that the invariance conditions ensuing from eq. (10.4) and from its antiunitary operator implementation are identical with the conditions (9.2) following from the C–type transformations (9.1) with the matrices R, U, U_H of eq. (10.4).[4] Clearly, if R, U, U_H are identity matrices, eq. (9.2) is always fulfilled suggesting that CPT should be defined by[5]

$$CPT: \qquad \begin{array}{rcl} W_\mu^a(x) & \to & -W_\mu^a(-x) \\ \omega_R(x) & \to & -\gamma_5^T \omega_R(-x)^* \\ \Phi(x) & \to & -\Phi(-x)^*. \end{array} \tag{10.5}$$

[4]We use the transposed matrices in eq. (10.4) to get exactly the form of eq. (9.2) for the invariance conditions.

[5]The minus in the transformation of ω_R is convention, however, the other two minus signs are required for invariance of the Lagrangian.

In other words, invariance of the Lagrangian under the transformation (10.4) leads to invariance under the corresponding C–type transformation (9.1) yet the Lagrangian is anyway invariant under the transformation (10.5). Thus the concept of CPT–type transformations is void and there is just one canonical form (10.5) of CPT. This together with the definition of CP eq. (5.1) via the contragredient automorphism $\psi^{\triangle}$ fixes the definition of time reversal in the generalized sense analogous to CP in sect. 5.

It is interesting to note that with $R = \mathbf{1}$, but U, U_H non–trivial, conditions (9.2) tell us that U, U_H act on ω_R, Φ, respectively, as representations of an element of $G \times H$. If U, U_H belong to H alone then the effect of the CPT–type transformation (10.4) or the corresponding C–type transformation (9.1) is a horizontal symmetry [39].

The comment on CPT explains why in the discussion of the discrete symmetries C, P, T we could concentrate on CP and P in this work and being fully general at the same time.

Let us now summarize the main points of this work. The starting point of our discussion was the observation that CP and P transformations have the same structure when formulated with fermion fields of one chirality. General transformations of that type we have called CP–type transformations. The crucial point is that if a CP–type transformation is a symmetry of the Lagrangian its action on the gauge bosons can be described in terms of automorphisms of the Lie algebra $\mathcal{L}_c$ of the gauge group. Consequently, the invariance conditions in the fermion and scalar sectors also have a straightforward interpretation in terms of Lie algebra representations. In addition, all possible automorphisms of simple Lie algebras are known and classified in the literature.

Now what distinguishes CP and P from each other? The automorphism associated with CP is given by the contragredient automorphism $\psi^{\triangle}$ which has the property $\psi^{\triangle}(h) = -h$ for all elements h of the CSA whereas P is associated with an involutive automorphism ψ_P which reverses the signs of at most part of the CSA. These abstract mathematical definitions were substantiated by physical considerations.

CP is always a symmetry of the gauge Lagrangian $\mathcal{L}_{\text{gauge}}$, the Lagrangian without Yukawa couplings and the Higgs potential. This statement formulated in the language of representations is expressed by $-D^T \circ \psi^{\triangle} \sim D$, saying that for every irrep D of $\mathcal{L}_c$ its complex conjugate irrep $-D^T$ connected with the automorphism $\psi^{\triangle}$ is equivalent to D. (This follows immediately from the fact that the weights of both irreps are identical.) Consequently, CP does not impose conditions on the irrep content of the fermion or scalar representation. If irreps occur with non–trivial multiplicities a CP transformation not acting in these "horizontal" spaces is called "canonical CP" or $(\mathrm{CP})_0$, a special case of the "generalized" CP transformations.

Since P is not uniquely associated with an automorphism the choice of ψ_P is subject to physical boundary conditions. For instance, one would require that ψ_P does not change the sign of those elements in the CSA which are associated with the electric and colour charges in order to get a reasonable definition of parity. Also one could imagine cases with large gauge groups where several automorphisms lead to viable definitions of P. In contrast to CP, the gauge interactions of fermions and scalars are not automatically invariant under parity but, in general, parity invariance introduces a condition on the irrep

content of the fermion and scalar representations or on irreps themselves, depending on the gauge group. For simple gauge groups a common condition is that with every irrep in the fermionic sector also its complex conjugate irrep occurs. This condition is always trivially satisfied in the scalar sector. We have also worked out the connection between parity and the definition of Dirac fields.

In the scalar sector a complication arises for pseudoreal irreps where the horizontal part of a CP–type transformation in general mixes the pseudoreal fields ϕ and $\widetilde{\phi} = W\phi^*$ (7.4) which transform alike under the gauge group. This leads to unitary symplectic matrices acting horizontally.

Though generalized CP transformations are automatically symmetries of $\mathcal{L}_{\mathrm{gauge}}$ this is not the case for the Yukawa interactions. Choosing a particular transformation one obtains conditions on the Yukawa couplings. For $(\mathrm{CP})_0$ this amounts to real couplings in some phase conventions. In the general case horizontal transformations are involved. With suitable basis choices all possible solutions of these conditions for all possible generalized CP transformations can be derived. This might be of interest for models relating fermion masses and mixing angles.

We have also considered C–type transformations defined as the composition of two CP–type transformations. Such symmetry transformations associated with the trivial automorphism $\psi_C = id$ are just horizontal symmetries. Finally we have shown that at least for simple gauge groups but arbitrary reps CP and P are compatible in the following sense: given fermion and scalar reps, real Yukawa couplings, i.e. $\mathcal{L}_Y$ invariant under $(\mathrm{CP})_0$, and an automorphism ψ_C associated with C such that $D \circ \psi_C$ is also contained in the reps for every irrep D occurring there, then one can construct Yukawa couplings which are invariant under the "simplest" C transformation pertaining to ψ_C.

In conclusion, in this work we have tried to understand CP and P symmetries in gauge theories by pointing out their intimate connection with automorphisms of the Lie algebra of the gauge group. We have furthermore studied in detail the action of such symmetries in the multiplicity spaces of the irreps in the fermionic and scalar sectors and how these rotations imply conditions on the Yukawa couplings. Finally, we have made extensive use of certain basis transformations: in the representation spaces of the irreps they lead to symmetric or antisymmetric generators of $\mathcal{L}_c$ in arbitrary irreps and real Clebsch Gordan coefficients in the Yukawa couplings and in the horizontal spaces they give simple forms of generalized CP transformations. All these considerations might be useful for the construction of models beyond the SM.

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Appendices

A Notation and conventions

We follow the conventions of ref. [15]. Therefore we use the metric

$$(g_{\mu\nu}) = \text{diag}\ (1, -1, -1, -1) \tag{A.1}$$

and thus the Dirac algebra is given by

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathbf{1}_4. \tag{A.2}$$

Furthermore, we assume that the γ matrices fulfill the hermiticity conditions

$$\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0 = \varepsilon(\mu)\gamma_\mu \quad \text{with} \quad \varepsilon(\mu) = \begin{cases} 1, & \mu = 0 \\ -1, & \mu = 1,2,3. \end{cases} \tag{A.3}$$

Then

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{A.4}$$

is hermitian. Left and right–handed fermion fields are given by the conditions

$$\frac{\mathbf{1}+\gamma_5}{2}\chi_R = \chi_R, \qquad \frac{\mathbf{1}-\gamma_5}{2}\chi_L = \chi_L, \tag{A.5}$$

respectively. The charge conjugation matrix C is defined by

$$C^{-1}\gamma_\mu C = -\gamma_\mu^T. \tag{A.6}$$

As a consequence of eqs. (A.3) and (A.6) we have[6]

$$C^\dagger = C^{-1}, \qquad C^T = -C \qquad \text{and} \qquad C^{-1}\gamma_5 C = \gamma_5^T. \tag{A.7}$$

A time reversal transformation requires the definition of a matrix T verifying

$$T\gamma_\mu T^{-1} = \gamma_\mu^T. \tag{A.8}$$

From the properties of C it is clear that

$$T = e^{i\beta}C^{-1}\gamma_5 = -T^T \tag{A.9}$$

with an arbitrary phase β. Then, given a solution $\chi(x)$ of the Dirac equation we obtain its time–reflected solution

$$\chi_T(x^0, \vec{x}) = T^*\chi(-x^0, \vec{x})^*. \tag{A.10}$$

[6] Actually, it can only be derived that $C^\dagger$ is proportional to C^{-1}. For convenience and without loss of generality we assume that these matrices are equal.

In the second quantized version this translates into

$$\mathcal{T}\chi(x^0,\vec{x})\mathcal{T}^{-1} = T\chi(-x^0,\vec{x}) \tag{A.11}$$

with the time reversal operator $\mathcal{T}$ acting on the Hilbert space of states as an antiunitary operator.

In this paper we only use right–handed fermion fields. If one starts in a theory with fields of both chirality f_L, f_R then the fermionic Lagrangian

$$\mathcal{L}_F = \bar{f}_R i\gamma^\mu(\partial_\mu + igT_a^R W_\mu^a)f_R + \bar{f}_L i\gamma^\mu(\partial_\mu + igT_a^L W_\mu^a)f_L, \tag{A.12}$$

where $\{T_a^R\}$, $\{T_a^L\}$ are arbitrary, in general different reps of the gauge group, can easily be rewritten as

$$\mathcal{L}_F = \bar{\omega}_R i\gamma^\mu(\partial_\mu + igT_a W_\mu^a)\omega_R \tag{A.13}$$

with

$$\omega_R = \begin{pmatrix} (f_L)^c \\ f_R \end{pmatrix}, \qquad (f_L)^c \equiv C\gamma_0^T f_L^* \qquad \text{and} \qquad T_a = \begin{pmatrix} -(T_a^L)^T & 0 \\ 0 & T_a^R \end{pmatrix}.$$

B Facts about semisimple Lie groups

In this appendix we collect all the facts about Lie algebras (in particular, semisimple Lie algebras) which are used in this work. Extensive expositions of this subject can e.g. be found in the books by Cornwell [16], Georgi [20], Samelson [24], Jacobson [25], Varadarajan [26], Wybourne [41], Cahn [42] and others.

The Killing form: Every element X of a Lie algebra (always assumed to be over a field $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$) allows to define a linear mapping

$$\text{ad } X : \begin{array}{ccc} \mathcal{L} & \rightarrow & \mathcal{L} \\ Y & \rightarrow & [X,Y]. \end{array} \tag{B.1}$$

Then the symmetric bilinear form κ obtained by

$$\kappa(X,Y) = \text{Tr } (ad\ X \text{ ad } Y) \tag{B.2}$$

is called Killing form. Automorphisms ψ of $\mathcal{L}$ are defined as linear mappings which respect the Lie algebra product, i.e.

$$\psi([X,Y]) = [\psi(X),\psi(Y)] \qquad \forall\ X,Y \in \mathcal{L}. \tag{B.3}$$

The set of automorphisms form a group denoted by Aut $(\mathcal{L})$. Note that the Killing form is invariant under automorphisms:

$$\kappa(\psi(X),\psi(Y)) = \kappa(X,Y)\ \forall\ X,Y \in \mathcal{L}. \tag{B.4}$$

Semisimple Lie algebras are those which have no non–zero Abelian ideals. Cartan's second criterion states that a Lie algebra $\mathcal{L}$ is semisimple if and only if its dimension is positive and its Killing form non–degenerate.

Given a basis $\{X_a\}$ of $\mathcal{L}$ one can define structure constants by

$$[X_a, X_b] = C^c_{ab} X_c. \tag{B.5}$$

One can show that a semisimple Lie group G (a group whose (real) Lie algebra is semisimple) is compact if and only if its Lie algebra $\mathcal{L}_c$ has a negative definite Killing form. Therefore on such a Lie algebra a scalar product is given by $-\kappa$. This allows the definition of ON bases $\{X_a\}$ with $\kappa(X_a, X_b) = -\delta_{ab}$. In such a case the structure constants C^c_{ab} are usually denoted by f_{abc}. One can prove that f_{abc} is totally antisymmetric in a, b, c whereas in general only $C^c_{ab} = -C^c_{ba}$ is valid.

Automorphisms and bases of $\mathcal{L}$: Fixing a basis $\{X_a\}$ of $\mathcal{L}$ allows to associate a matrix with every linear operator on $\mathcal{L}$ and vice versa. Denoting such a matrix by A and the corresponding operator by ψ_A we can thus write

$$\psi_A : \begin{array}{ccc} \mathcal{L} & \to & \mathcal{L} \\ X_a & \to & A_{ba} X_b. \end{array} \tag{B.6}$$

It is then easy to check by analysing eqs. (B.3) and (B.5) that matrices A corresponding to automorphisms ψ_A fulfill the following condition:

$$\psi_A \in \text{Aut } (\mathcal{L}) \Longleftrightarrow A_{a'a} A_{b'b} (A^{-1})_{cc'} C^{c'}_{a'b'} = C^c_{ab}. \tag{B.7}$$

Furthermore, invariance of the Killing form is expressed by

$$A_{a'a} A_{b'b} \kappa(X_{a'}, X_{b'}) = \kappa(X_a, X_b). \tag{B.8}$$

For compact Lie algebras and ON bases this immediately translates into the conditions that A is an orthogonal matrix and f_{abc} an invariant tensor with respect to A:

$$A_{a'a} A_{b'b} A_{c'c} f_{a'b'c'} = f_{abc}. \tag{B.9}$$

The structure of semisimple Lie algebras: With every Lie group G a real Lie algebra $\mathcal{L}$ is associated and by complexification of $\mathcal{L}$ a complex Lie algebra $\widetilde{\mathcal{L}}$. The transition from $\mathcal{L}$ to $\widetilde{\mathcal{L}}$ conserves semisimplicity. Complex semisimple Lie algebras are fully classified. Conversely, given a complex semisimple Lie algebra $\widetilde{\mathcal{L}}$ there is a well defined procedure to go back to the real Lie algebras $\mathcal{L}$ associated with $\widetilde{\mathcal{L}}$, i.e. to find all inequivalent $\mathcal{L}$'s whose complexification is $\widetilde{\mathcal{L}}$. The structure of $\mathcal{L}$ is closely connected with $\widetilde{\mathcal{L}}$. This motivates the consideration of complex semisimple Lie algebras though in gauge theories only real Lie algebras occur. Furthermore, since here we are concerned only with compact Lie groups it is sufficient to consider the construction of the unique compact Lie

algebra $\mathcal{L}_c$ associated with $\widetilde{\mathcal{L}}$ (see eq. (B.23)) and refer the reader to the books quoted at the beginning of this appendix for the general procedure of "realification".

A CSA can be defined and its existence proved for general Lie algebras (see refs. [24, 25, 26]). In the case of a semisimple complex Lie algebra a CSA $\mathcal{H}$ is a maximal abelian subalgebra of $\widetilde{\mathcal{L}}$ such that ad h is completely reducible (i.e. diagonalizable) $\forall\, h \in \mathcal{H}$. All CSAs are mutually conjugate which means that given two CSAs $\mathcal{H}$, $\mathcal{H}'$ of $\widetilde{\mathcal{L}}$ then there is an inner automorphism (see eq. (4.1)) ψ such that $\psi(\mathcal{H}) = \mathcal{H}'$. In this sense a CSA is unique for semisimple complex Lie algebras $\widetilde{\mathcal{L}}$. The dimension of the CSA, $\dim \mathcal{H} = \ell$, is called the rank of $\widetilde{\mathcal{L}}$.

As a vector space $\widetilde{\mathcal{L}}$ can be decomposed into

$$\widetilde{\mathcal{L}} = \left(\bigoplus_{\alpha\in\Delta} \widetilde{\mathcal{L}}_\alpha \right) \oplus \mathcal{H} \tag{B.10}$$

which corresponds to the decomposition of $\widetilde{\mathcal{L}}$ into common eigenstates of ad h ($h \in \mathcal{H}$):

$$(\mathrm{ad}\, h)(X) = [h, X] = \alpha(h) X \tag{B.11}$$

with $X \in \widetilde{\mathcal{L}}_\alpha$. One can show that

$$\dim \widetilde{\mathcal{L}}_\alpha = 1 \qquad \forall\, \alpha \in \Delta \tag{B.12}$$

for $\widetilde{\mathcal{L}}$ semisimple. The eigenvalues $\alpha(h)$ depend linearly on h and can therefore be regarded as linear functionals on $\mathcal{H}$. These non–zero functionals are called roots and the set of roots is denoted by Δ. Since the spaces $\widetilde{\mathcal{L}}_\alpha$ are one–dimensional it suffices to choose a non–zero vector $e_\alpha \in \widetilde{\mathcal{L}}_\alpha$ as a basis $\forall\, \alpha \in \Delta$. Then one can show that

$$\kappa(e_\alpha, h) = 0 \quad \forall\, \alpha \in \Delta,\ h \in \mathcal{H} \quad \text{and} \quad \kappa(e_\alpha, e_\beta) = 0 \quad \forall\, \alpha, \beta \in \Delta \text{ with } \beta \neq -\alpha. \tag{B.13}$$

Therefore, non–degeneracy of κ requires

$$\alpha \in \Delta \Longleftrightarrow -\alpha \in \Delta. \tag{B.14}$$

As a matter of fact already $\kappa|_{\mathcal{H}}$ is non–degenerate. Thus $\forall\, \alpha \in \Delta$ there exists a unique $h_\alpha \in \mathcal{H}$ such that

$$\alpha(h) = \kappa(h_\alpha, h) \qquad \forall\, h \in \mathcal{H}. \tag{B.15}$$

The elements h_α are called root vectors. If $\alpha, \beta, \alpha+\beta \in \Delta$ then

$$h_{\alpha+\beta} = h_\alpha + h_\beta, \qquad h_{-\alpha} = h_\alpha. \tag{B.16}$$

The real linear span of $\{h_\alpha | \alpha \in \Delta\}$ is denoted by $\mathcal{H}_{\mathbf{R}}$ and $\mathcal{H}$ is the complexification of $\mathcal{H}_{\mathbf{R}}$. $\kappa|_{\mathcal{H}_{\mathbf{R}}}$ is a positive definite scalar product and therefore $\mathcal{H}_{\mathbf{R}}$ is an ℓ–dimensional euclidean space. By

$$\langle \alpha, \beta \rangle \equiv \kappa(h_\alpha, h_\beta)\, \forall \alpha, \beta \in \Delta \tag{B.17}$$

length of roots and the angle between two roots are defined. The basis elements e_α and the root vectors are connected through

$$[e_\alpha, e_{-\alpha}] = \kappa(e_\alpha, e_{-\alpha}) h_\alpha. \tag{B.18}$$

A crucial point of the theory is that the numbers

$$a_{\beta\alpha} \equiv 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \tag{B.19}$$

are integers (the Cartan integers) and that only $a_{\beta\alpha} = 0, \pm 1, \pm 2, \pm 3$ is allowed.

On Δ a weak order can be defined by choosing an element $h_0 \in \mathcal{H}_{\mathbf{R}}$ such that $\alpha(h_0) \neq 0 \ \forall \ \alpha \in \Delta$. Then for functionals μ, μ' on $\mathcal{H}_{\mathbf{R}}$ the relation $\mu > \mu'$ ($\mu \geq \mu'$) is defined by $\mu(h_0) > \mu'(h_0)$ ($\mu(h_0) \geq \mu'(h_0)$). Let $\Delta_\pm = \{\alpha \in \Delta | \alpha(h_0) \gtrless 0\}$. Then $\Delta = \Delta_+ \cup \Delta_-$ and $\Delta_- = \{-\alpha | \alpha \in \Delta_+\}$. A root is called simple if it is positive but not the sum of two positive roots. The set of simple roots consists of exactly ℓ linearly independent elements $\{\alpha_1, \ldots, \alpha_\ell\}$.

The Weyl canonical form of $\widetilde{\mathcal{L}}$: By choosing suitable bases the commutation relations characterizing semisimple complex Lie algebras can be brought to certain standard forms one of which is the Weyl canonical form (for other standard forms see ref. [16]). There the basis elements e_α are normalized to

$$\kappa(e_\alpha, e_{-\alpha}) = -1. \tag{B.20}$$

Then one has the following commutators:

$$\begin{array}{rcl} [e_\alpha, e_{-\alpha}] &=& -h_\alpha \\ [h, e_\alpha] &=& \alpha(h) e_\alpha \qquad \text{or} \qquad [h_\beta, e_\alpha] = \langle \beta, \alpha \rangle e_\alpha \\ [h, h'] &=& 0 \qquad \forall\, h, h' \in \mathcal{H} \\ [e_\alpha, e_\beta] &=& 0 \qquad \text{if } \alpha + \beta \notin \Delta,\ \alpha + \beta \neq 0 \\ [e_\alpha, e_\beta] &=& N_{\alpha\beta} e_{\alpha+\beta} \qquad \text{for } \alpha + \beta \in \Delta \end{array} \tag{B.21}$$

with

$$N_{\alpha\beta} \in \mathbf{R} \setminus \{0\}, \qquad N_{\alpha\beta} = N_{-\alpha-\beta}.$$

Given a weak order on Δ every root can be written as a linear combination of simple roots such that $\forall\ \alpha \in \Delta_+$

$$\alpha = \sum_{j=1}^{\ell} k_j^\alpha \alpha_j \qquad \text{with} \qquad k_j^\alpha \in \mathbf{N}_0 \tag{B.22}$$

for rank $\ell = \dim \mathcal{H} = \dim_{\mathbf{R}} \mathcal{H}_{\mathbf{R}}$.

The compact real form $\mathcal{L}_c$ of $\widetilde{\mathcal{L}}$: Given $\widetilde{\mathcal{L}}$ we can go back to the compact real Lie algebra $\mathcal{L}_c$ by choosing the following basis elements $\{X_a\}$:

$$
\begin{gathered}
-iH_j \ (j = 1, \ldots, \ell) \qquad \text{with} \qquad H_j \in \mathcal{H}_{\mathbf{R}}, \qquad \kappa(H_j, H_k) = \delta_{jk}, \\
\frac{e_\alpha + e_{-\alpha}}{\sqrt{2}} \qquad \text{and} \qquad \frac{e_\alpha - e_{-\alpha}}{\sqrt{2}\, i} \qquad \forall\, \alpha \in \Delta_+.
\end{gathered} \tag{B.23}
$$

To a given semisimple complex Lie algebra $\widetilde{\mathcal{L}}$ there is a unique compact real form $\mathcal{L}$, up to isomorphisms. One can easily check that the above basis elements fulfill $\kappa(X_a, X_b) = -\delta_{ab}$.

Dynkin diagrams: With the simple roots the Cartan matrix

$$
A_{jk} = 2\frac{\langle \alpha_j, \alpha_k \rangle}{\langle \alpha_k, \alpha_k \rangle} \tag{B.24}
$$

is associated. Clearly, $A_{jj} = 2$ and one can show that A_{jk} can only be $0, -1, -2$ or -3.

Considering the case $j \neq k$ in more detail we find (no sum implied here)

$$
A_{jk} A_{kj} = 4 \cos^2 \Theta \tag{B.25}
$$

where Θ is the angle between α_j and α_k. Note that for $\Theta \neq \pi/2$ either A_{jk} or A_{kj} have to be -1. According to the possible angles Θ we have to distinguish four cases (assuming that $A_{jk} = -1$ for $\Theta \neq \pi/2$ and $\omega_j^2 \equiv \langle \alpha_j, \alpha_j \rangle$):

a) $\cos\Theta = 0 \Rightarrow \Theta = \pi/2$ or 90^0, ω_k/ω_j undetermined

b) $\cos\Theta = -1/2 \Rightarrow \Theta = 2\pi/3$ or 120^0, $\omega_k/\omega_j = 1$

c) $\cos\Theta = -1/\sqrt{2} \Rightarrow \Theta = 3\pi/4$ or 135^0, $\omega_k/\omega_j = \sqrt{2}$

d) $\cos\Theta = -\sqrt{3}/2 \Rightarrow \Theta = 5\pi/6$ or 150^0, $\omega_k/\omega_j = \sqrt{3}$.

For $\Theta \neq 0$ and $A_{jk} = -1$ the ratio of the lengths of the simple roots is given by $\omega_k/\omega_j = -2\cos\Theta$.

A Dynkin diagram of a semisimple complex Lie algebra is defined in the following way:

i) To each simple root is associated a point (or vertex) of the diagram.

ii) The points associated with α_j and α_k are connected by $A_{jk}A_{kj}$ lines, i.e. there are zero, one, two or three lines for the above cases a, b, c, d, respectively.

iii) If there are two or three lines connecting two points then a black dot denotes the shorter root. Thus a Dynkin diagram is the graphical representation of the Cartan matrix.

The connected Dynkin diagrams are exactly those associated with simple complex Lie algebras and there is a one–to–one correspondence between connected Dynkin diagrams and simple complex Lie algebras. In fig. 1 all possible connected Dynkin diagrams are depicted with the names of the $\widetilde{\mathcal{L}}$'s associated with them. In table 1 all $\widetilde{\mathcal{L}}$'s are listed with their respective compact real forms $\mathcal{L}_c$. Table 2 contains all isomorphisms of the low–dimensional classical $\mathcal{L}_c$'s explaining thus why the series B_ℓ, C_ℓ, D_ℓ start with $\ell = 2, 3, 4$, respectively.

Some facts about irreps of semisimple complex Lie algebras: All facts mentioned here are also valid for $\mathcal{L}_c$.

In a rep of $\widetilde{\mathcal{L}}$ weight vectors are those elements of the vector space which are eigenvectors of the CSA, i.e.

$$D(h)e(\lambda, q) = \lambda(h)e(\lambda, q) \qquad (q = 1, \ldots, m_\lambda) \tag{B.26}$$

where m_λ is the multiplicity of the weight λ which, like a root, can be conceived as a functional on $\mathcal{H}$. One can show that for any weight λ of a rep of $\widetilde{\mathcal{L}}$ and for any $\alpha \in \Delta$

$$\frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbf{Z} \quad \text{and} \quad \lambda = \sum_{j=1}^{\ell} \mu_j \alpha_j \text{ with } \mu_j \text{ real and rational.} \tag{B.27}$$

The fundamental weights are defined by

$$\Lambda_j = \sum_{k=1}^{\ell} (A^{-1})_{jk} \alpha_k \tag{B.28}$$

and it follows that

$$\frac{2\langle \Lambda_j, \alpha_k \rangle}{\langle \alpha_k, \alpha_k \rangle} = \delta_{jk}. \tag{B.29}$$

For any irrep of a semisimple complex Lie algebra $\widetilde{\mathcal{L}}$ there is a unique highest weight Λ (a weight Λ is called highest if $\Lambda + \alpha$ is not a weight $\forall \alpha \in \Delta_+$). It can be written as

$$\Lambda = n_1 \Lambda_1 + \ldots + n_\ell \Lambda_\ell \tag{B.30}$$

where $n_1, \ldots, n_\ell$ are non–negative integers. Conversely, given $n_1, \ldots, n_\ell$ with all $n_j \in \mathbf{N}_0$ there is an irrep D_Λ of $\widetilde{\mathcal{L}}$ unique up to equivalence such that $\Lambda = n_1\Lambda_1 + \ldots + n_\ell\Lambda_\ell$ is its highest weight. Thus there is a one–to–one correspondence between functionals Λ of the form (B.30) and irreps D_Λ of $\widetilde{\mathcal{L}}$. All irreps are faithful except the trivial irrep with $\Lambda = 0$.

No real form $\mathcal{L}$ of $\widetilde{\mathcal{L}}$ admits unitary non–trivial irreps except the compact real form $\mathcal{L}_c$ where all irreps are equivalent to unitary ones. On $D_\Lambda|_{\mathcal{L}_c}$ unitarity is expressed by

$$D(X)^\dagger = -D(X) \tag{B.31}$$

corresponding to unitary operators $\exp D(X)$. Then from the basis (B.23) we derive that

$$D(H)^\dagger = D(H) \quad \forall\, H \in \mathcal{H}_{\mathbf{R}}, \qquad D(e_\alpha)^\dagger = -D(e_{-\alpha}) \quad \forall\, \alpha \in \Delta. \tag{B.32}$$

Finally we want to mention that in physics one usually employs hermitian generators of a unitary rep

$$T_a^\dagger = T_a, \qquad [T_a, T_b] = i f_{abc} T_c \tag{B.33}$$

which are obtained by

$$T_a = i D(X_a) \qquad \text{with} \qquad [D(X_a), D(X_b)] = f_{abc} D(X_c) \tag{B.34}$$

from the ON generators X_a of $\mathcal{L}_c$. In this paper we often switch between the two forms $\{T_a\}$ and $\{D(X_a)\}$.

C $\quad so(N)$ and the spinor representations

A convenient choice of basis in the space of antisymmetric real $N \times N$ matrices is given by

$$(M_{pq})_{jk} = \delta_{pj}\delta_{qk} - \delta_{qj}\delta_{pk}, \qquad 1 \le p < q \le N \tag{C.1}$$

with commutation relations

$$[M_{pq}, M_{rs}] = \delta_{ps} M_{qr} + \delta_{qr} M_{ps} - \delta_{pr} M_{qs} - \delta_{qs} M_{pr}. \tag{C.2}$$

A Clifford algebra with N basis elements is defined by the anticommutators

$$\{\Gamma_p, \Gamma_q\} = 2\delta_{pq}\mathbf{1}. \tag{C.3}$$

It is easy to show that the elements

$$\frac{1}{2}\sigma_{pq} \equiv \frac{1}{4}[\Gamma_p, \Gamma_q] \tag{C.4}$$

also verify the commutation relations given by eq. (C.2). Thus a rep of the Clifford algebra automatically gives also a rep of $so(N)$. It is straightforward to check that for $N = 2\ell + 1$ a rep of the $\{\Gamma_p\}$ in the 2^ℓ dimensional space $\mathbf{C}^2 \otimes \ldots \otimes \mathbf{C}^2$ (ℓ–fold tensor product) is given, in terms of the Pauli matrices and the two–dimensional identity matrix, by

$$\begin{array}{rclcrcl}
\Gamma_1 & = & \sigma_3 \otimes \ldots \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_1 & \qquad & \Gamma_{2\ell-3} & = & \sigma_3 \otimes \sigma_1 \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1} \\
\Gamma_2 & = & \sigma_3 \otimes \ldots \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_2 & & \Gamma_{2\ell-2} & = & \sigma_3 \otimes \sigma_2 \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1} \\
\Gamma_3 & = & \sigma_3 \otimes \ldots \otimes \sigma_3 \otimes \sigma_1 \otimes \mathbf{1} & & \Gamma_{2\ell-1} & = & \sigma_1 \otimes \mathbf{1} \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1} \\
\Gamma_4 & = & \sigma_3 \otimes \ldots \otimes \sigma_3 \otimes \sigma_2 \otimes \mathbf{1} & & \Gamma_{2\ell} & = & \sigma_2 \otimes \mathbf{1} \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1} \\
 & \vdots & & & \Gamma_{2\ell+1} & = & \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \ldots \otimes \sigma_3.
\end{array} \tag{C.5}$$

Then the $\{\frac{1}{2}\sigma_{pq}\}$ of eq. (C.4) define the irreducible spinor irrep of $so(2\ell+1)$ which has dimension 2^ℓ.

For $so(2\ell)$ one simply has to take $\Gamma_1, \ldots, \Gamma_{2\ell}$ of the Clifford algebra of $so(2\ell+1)$. However, the 2^ℓ–dimensional rep of $so(2\ell)$ is not irreducible because

$$[\sigma_{pq}, \Gamma_{2\ell+1}] = 0 \qquad \forall\, p, q = 1, \ldots, \ell. \tag{C.6}$$

It decays into two inequivalent irreps of dimension $2^{\ell-1}$ given by the projectors $(\mathbf{1} \pm \Gamma_{2\ell+1})/2$.

D On the isomorphisms $so(4) \cong su(2) \oplus su(2)$ and $so(6) \cong su(4)$

In sect. 2 the above isomorphisms are exploited for the example of the spinor representation of $so(10)$. Since there one assumes that it is known how to classify the fields according to $su(4) \oplus su(2) \oplus su(2)$ an explicit realization of the above isomorphisms has to be established to transfer this classification to $so(6) \oplus so(4) \subset so(10)$.

Starting from the basis $\{M_{ij}\}$ as given in app. C we can define the following new basis of $so(4)$:

$$\begin{aligned} A_1 &= \frac{1}{2}(M_{23} - M_{14}) \qquad & B_1 &= \frac{1}{2}(M_{23} + M_{14}) \\ A_2 &= \frac{1}{2}(M_{13} - M_{42}) \qquad & B_2 &= \frac{1}{2}(M_{13} + M_{42}) \\ A_3 &= \frac{1}{2}(M_{12} - M_{34}) \qquad & B_3 &= \frac{1}{2}(M_{12} + M_{34}). \end{aligned} \tag{D.1}$$

Then it is easy to check with eqs. (C.1) and (C.2) that

$$[A_i, A_j] = \varepsilon_{ijk} A_k, \qquad [B_i, B_j] = \varepsilon_{ijk} B_k, \qquad [A_i, B_j] = 0 \tag{D.2}$$

thus proving the first of the isomorphisms.

In order to prove the second isomorphism we will specify a basis $\{E_a\}$ of $so(6)$ such that there is a one–to–one correspondence between the matrices $\{E_a\}$ and the matrices $\{\lambda_a\}$ of $su(4)$ obtained by generalizing the Gell–Mann basis of $su(3)$, i.e.

$$-\, i\frac{\lambda_a}{2} \longleftrightarrow E_a \qquad (a = 1, \ldots, 15), \tag{D.3}$$

such that the matrices E_a obey the same commutation relations as $-i\lambda_a/2$. The result

which is unique up to orthogonal basis transformations is given by[7]

$$
\begin{aligned}
E_1 &= \frac{1}{2}(M_{23}-M_{14}) & E_9 &= \frac{1}{2}(M_{45}+M_{36}) \\
E_2 &= \frac{1}{2}(M_{13}-M_{42}) & E_{10} &= -\frac{1}{2}(M_{35}+M_{64}) \\
E_3 &= \frac{1}{2}(M_{12}-M_{34}) & E_{11} &= \frac{1}{2}(M_{16}+M_{25}) \\
E_4 &= \frac{1}{2}(M_{16}-M_{25}) & E_{12} &= -\frac{1}{2}(M_{15}+M_{62}) \\
E_5 &= \frac{1}{2}(M_{62}-M_{15}) & E_{13} &= \frac{1}{2}(M_{23}+M_{14}) \\
E_6 &= \frac{1}{2}(M_{45}-M_{36}) & E_{14} &= -\frac{1}{2}(M_{13}+M_{42}) \\
E_7 &= \frac{1}{2}(M_{35}-M_{64}) & E_{15} &= -\frac{1}{\sqrt{6}}(M_{12}+M_{34}+M_{56}) \\
E_8 &= \frac{1}{\sqrt{12}}(M_{12}+M_{34}-2M_{56}). & &
\end{aligned}
\tag{D.4}
$$

E Irreps of G^*

For the definition of G^* see eq. (4.10). Given a rep of G^* it can be decomposed according to the subgroup $G \times G$. Since irreps of $G \times G$ are given by $D_r(g_1) \otimes D_{r'}(g_2)$ ($(g_1, g_2) \in G \times G$) with D_r, $D_{r'}$ being irreps of G [30] we infer that an irrep D of G^* decays into $\bigoplus_{(r,r')}(D_r(g_1) \otimes D_{r'}(g_2))$ under $G \times G$. Using

$$D(E)D((g_1, g_2))D(E) = D((g_2, g_1)) \tag{E.1}$$

we obtain

$$\bigoplus_{(r,r')}(D_r(g_1) \otimes D_{r'}(g_2)) \sim \bigoplus_{(r,r')}(D_r(g_2) \otimes D_{r'}(g_1)) \sim \bigoplus_{(r,r')}(D_{r'}(g_1) \otimes D_r(g_2)). \tag{E.2}$$

Therefore, in the irrep D only summands of the type

$$D_r(g_1) \otimes D_r(g_2) \quad \text{and} \quad (D_r(g_1) \otimes D_{r'}(g_2)) \oplus (D_{r'}(g_1) \otimes D_r(g_2)) \quad (r \neq r') \tag{E.3}$$

appear and thus

$$
\begin{aligned}
D((g_1, g_2)) \sim & \left[\bigoplus_r m_r (D_r(g_1) \otimes D_r(g_2))\right] \\
& \oplus \left[\bigoplus_{(r',r'')} m_{r',r''}((D_{r'}(g_1) \otimes D_{r''}(g_2)) \oplus (D_{r''}(g_1) \otimes D_{r'}(g_2)))\right]
\end{aligned}
\tag{E.4}
$$

[7]The first eight matrices listed are generators of the $su(3)$ subalgebra and can be obtained by a general procedure to derive the generators of the $su(N)$ subalgebra of $so(2N)$ as given in ref. [20].

with $r' \neq r''$ and m_r, $m_{r',r''}$ being the multiplicities of the reps (E.3). The associated vector spaces will be denoted by $\mathcal{V}_r$ and $\mathcal{V}_{r',r''}$, respectively. Now we can define an operator S by

$$S: \begin{array}{ccccl} x \otimes y & \to & y \otimes x & \text{on } \mathcal{V}_r, \\ (v \otimes w, x \otimes y) & \to & (y \otimes x, w \otimes v) & \text{on } \mathcal{V}_{r',r''}. \end{array} \tag{E.5}$$

One can easily prove that

$$SD((g_1, g_2))S = D((g_2, g_1)) \tag{E.6}$$

and therefore

$$[D(E)S, D((g_1, g_2))] = 0. \tag{E.7}$$

Schur's lemma guarantees that $D(E)S$ operates within $\mathcal{V}_r^{\oplus m_r}$ and $\mathcal{V}_{r',r''}^{\oplus m_{r',r''}}$ (the superscript $\oplus m$ denotes the m–fold direct sum).

Discussing first $\mathcal{V}_r^{\oplus m_r} \cong \mathbf{C}^{m_r} \otimes \mathcal{V}_r$ we note that

$$D(E) = A \otimes S \qquad \text{with} \qquad A^2 = \mathbf{1}_{m_r} \tag{E.8}$$

since $S^2 = id$. Therefore we can make a basis transformation in $\mathbf{C}^{m_r}$ diagonalizing A with eigenvalues ± 1. Consequently we have a type of irreps given by

$$\mathcal{V}_r \qquad \text{and} \qquad D(E) = \pm S. \tag{E.9}$$

This defines the irreps $D_r^{\pm}$ according to the sign in eq. (E.9).

Writing $\mathcal{V}_{r',r''} = \mathcal{W}_{r',r''} \oplus \mathcal{W}_{r'',r'}$ according to the two inequivalent irreps of $G \times G$ on $\mathcal{V}_{r',r''}$ and $\mathcal{V}_{r',r''}^{\oplus m_{r',r''}} \cong (\mathbf{C}^{m_{r',r''}} \otimes \mathcal{W}_{r',r''}) \oplus (\mathbf{C}^{m_{r',r''}} \otimes \mathcal{W}_{r'',r'})$ then $D(E)$ has the form

$$D(E) = (A \otimes id_{r',r''}, B \otimes id_{r'',r'})S. \tag{E.10}$$

A small calculation reveals that $D(E)^2 = id$ requires

$$B = A^{-1}. \tag{E.11}$$

Performing a basis transformation with

$$Z = (\mathbf{1}_{m_{r',r''}} \otimes id_{r',r''}, A^{-1} \otimes id_{r'',r'}) \tag{E.12}$$

we obtain

$$Z^{-1}D((g_1, g_2))Z = D((g_1, g_2)) \qquad \text{and} \qquad Z^{-1}D(E)Z = S \tag{E.13}$$

on $\mathcal{V}_{r',r''}^{\oplus m_{r',r''}}$. Therefore the rep of G^* on this space decays into $m_{r',r''}$ equivalent copies of an irrep denoted by $D_{r',r''}$ defined via $D(E) = S$ as given in the second line of eq. (E.5). Thus we have found all irreps of G^* as written down in subsect. 4.2.

F On the existence of a CP basis

Let $\widetilde{\mathcal{L}}$ be a complex semisimple Lie algebra with a d–dimensional irrep D and $D(X_a) \equiv Y_a$ ($a = 1, \ldots, n_G$) where $\{X_a\}$ is an ON basis of the compact real form $\mathcal{L}_c$ of $\widetilde{\mathcal{L}}$. Thus the Y_a are $d \times d$ matrices and since D is unitary we have $Y_a^\dagger = -Y_a$ in addition.

In this appendix we will show that one can always choose a basis of $\mathbf{C}^d$ such that

$$Y_a^T = -\eta_a Y_a \tag{F.1}$$

in this basis and where the signs η_a are given by $\psi^\triangle(X_a) = \eta_a X_a$ with $\psi^\triangle$ being the contragredient automorphism defined in eq. (4.5). For the connection between the signs η_a and the ON basis $\{X_a\}$ defined in eq. (B.23) see eq. (5.12). Eq. (F.1) defines the $(\mathrm{CP})_0$ basis used in subsect. 5.2.

Proof: We have

$$-D^T \circ \psi^\triangle \sim D \tag{F.2}$$

because the weights of both irreps are identical. This means that there is a unitary matrix U such that

$$U(-Y_a^T \eta_a)U^\dagger = Y_a \tag{F.3}$$

and, consequently,

$$[UU^*, Y_a] = 0 \qquad \forall\, a = 1, \ldots, n_G. \tag{F.4}$$

Using Schur's lemma we obtain $UU^* = \lambda \mathbf{1}$ and therefore

$$U^T = \lambda U \qquad \text{with} \qquad \lambda = \pm 1. \tag{F.5}$$

Thus we have shown that the matrix U must be either symmetric or antisymmetric. (Exactly the same reasoning is valid for the matrix in the equivalence $-D^T \sim D$ in which case $\lambda = 1$ corresponds to real and $\lambda = -1$ to pseudoreal irreps.)

Case 1: $U = U^T$

We use the following lemma: For every unitary symmetric matrix U there is a unitary symmetric matrix $\widetilde{U}$ such that $U = \widetilde{U}^2$. Its proof given at the end of this appendix.

Inserting $\widetilde{U}^2$ into eq. (F.3) we obtain

$$-\eta_a \widetilde{U} Y_a^T \widetilde{U}^\dagger = -\eta_a (\widetilde{U}^\dagger Y_a \widetilde{U})^T = \widetilde{U}^\dagger Y_a \widetilde{U} \tag{F.6}$$

which is just the desired result (F.1).

Case 2: $U^T = -U$

It remains to show that this case is impossible for irreps of semisimple Lie algebras $\mathcal{L}_c$.

The irrep D is characterized by its highest weight Λ which is simple. We get from eq. (F.3) particularized for $D(-iH_j)$

$$U\left(D(-iH_j)^T\right)U^\dagger e_\Lambda = \Lambda(-iH_j)e_\Lambda \tag{F.7}$$

where e_Λ is the weight vector corresponding to Λ. Using antisymmetry of U and the fact that $D(-iH_j)$ is antihermitian eq. (F.7) can be rewritten as

$$D(-iH_j)Ue^*_\Lambda = \Lambda(-iH_j)Ue^*_\Lambda. \tag{F.8}$$

Since Λ is simple we must have $Ue^*_\Lambda = ae_\Lambda$ with $a \neq 0$. Finally, using $U^T = -U$ we derive the contradiction

$$0 = e^\dagger_\Lambda U e^*_\Lambda = a e^\dagger_\Lambda e_\Lambda \neq 0. \tag{F.9}$$

Thus U is symmetric and therefore the existence of a CP basis for any irrep of any compact semisimple Lie algebra is proved. □

Proof of the lemma: Let A be a normal symmetric matrix and $A_1 \equiv \mathrm{Re}\, A$, $A_2 \equiv \mathrm{Im}\, A$. Then it follows from $A^\dagger A = AA^\dagger$ and $A^T = A$ that $[A_1, A_2] = 0$. Thus A can be diagonalized by an orthogonal matrix O.

Let now U be unitary and symmetric. Then

$$U = O\hat{d}O^T \qquad \text{with} \qquad \hat{d} = \mathrm{diag}\,(e^{i\Omega_1}, \dots, e^{i\Omega_d})$$

and

$$\widetilde{U} = O\,\sqrt{\hat{d}}\,O^T \qquad \text{with} \qquad \sqrt{\hat{d}} = \mathrm{diag}\,(e^{i\Omega_1/2}, \dots, e^{i\Omega_d/2}).$$

□

G Basis transformations for pseudoreal scalars

In a situation with m scalar multiplets transforming under the group according to a pseudoreal irrep D we are led to matrices $H \in Sp(2m)$ in the CP–type transformation (7.8). Basis transformations of the form (3.26) involve again $Sp(2m)$ matrices. This suggests the question whether for a given $H \in Sp(2m)$ one can find a $Z \in Sp(2m)$ such that

$$H' = Z^\dagger H Z^* \tag{G.1}$$

is as "simple" as possible. In eq. (8.15) we have written down such a normal form. The proof of its existence will be given here.

It is useful to reformulate the problem in terms of antilinear operators and then take advantage of their properties. Let us define the antilinear operator

$$K : x \to x^*, \tag{G.2}$$

the matrix

$$J_m = \begin{pmatrix} 0 & \mathbf{1}_m \\ -\mathbf{1}_m & 0 \end{pmatrix} \tag{G.3}$$

and the antilinear operators

$$A \equiv HK, \qquad B \equiv J_m K \tag{G.4}$$

on $\mathbf{C}^{2m}$. Then the symplectic property of H is expressed as

$$[A, B] = 0. \tag{G.5}$$

Further properties of A, B are

$$A^\dagger A = \mathbf{1}, \qquad B^\dagger B = \mathbf{1} \qquad \text{and} \qquad B^2 = -\mathbf{1}. \tag{G.6}$$

This allows the following reformulation of the problem posed above (see eq. (G.1)):

Given two antilinear operators A, B on a unitary space $\mathcal{V}$ with properties (G.5) and (G.6) can one find canonical forms for A, B? The answer is the following.

Theorem: One can always find an ON basis such that B is given by $J_m K$ and A by eq. (8.15) in that basis. Since basis transformations for matrices corresponding to antilinear operators are performed according to eq. (G.1) the original problem is solved as well.

Proof: A is antiunitary and therefore A^2 is unitary. Let λ be an eigenvalue of A^2 with eigenvector v. Then Av has eigenvalue λ^*. Therefore the eigenvalues of A^2 can be denoted by

$$\lambda_1, \ldots, \lambda_\nu, \lambda_1^*, \ldots, \lambda_\nu^*, 1, -1 \qquad \text{with} \qquad |\lambda_i| = 1,\ \lambda_i \neq \pm 1\ (i = 1, \ldots, \nu) \tag{G.7}$$

with degeneracies $m_1, \ldots, m_\nu$ and $m_\pm$, respectively. The decomposition of $\mathcal{V}$ into eigenspaces of A^2 is given by

$$\mathcal{V} = \bigoplus_{i=1}^{\nu} (\mathcal{V}(\lambda_i) \oplus \mathcal{V}(\lambda_i^*)) \oplus \mathcal{V}_+ \oplus \mathcal{V}_-$$

with

$$A\mathcal{V}(\lambda_i) = \mathcal{V}(\lambda_i^*), \qquad B\mathcal{V}(\lambda_i) = \mathcal{V}(\lambda_i^*), \qquad A\mathcal{V}_\pm = \mathcal{V}_\pm, \qquad B\mathcal{V}_\pm = \mathcal{V}_\pm. \tag{G.8}$$

Therefore the proof of the above theorem can be devided into three parts.

a) $\mathcal{V} = \mathcal{V}(\lambda) \oplus \mathcal{V}(\lambda^*)$, $\lambda \neq \pm 1$, $\dim \mathcal{V}(\lambda) = \dim \mathcal{V}(\lambda^*) = m_\lambda$

We choose an ON basis $\{e_1, \ldots, e_{m_\lambda}\}$ in $\mathcal{V}(\lambda)$. Then $\{f_i = -Be_i |\ i = 1, \ldots, m_\lambda\}$ defines an ON basis in $\mathcal{V}(\lambda^*)$. This allows to write

$$Ae_i = M_{ji} f_j, \qquad Af_i = \widetilde{M}_{ji} e_j. \tag{G.9}$$

Antiunitarity of A results in

$$\delta_{ij} = \langle Ae_i | Ae_j \rangle = M_{ki}^* M_{kj} \tag{G.10}$$

with an analogous consideration for $\widetilde{M}$. Consequently, M, $\widetilde{M}$ are unitary $m_\lambda \times m_\lambda$ matrices. Furthermore, we infer from eq. (G.9) that

$$A^2 e_i = \lambda e_i = (\widetilde{M} M^*)_{ki} e_k. \tag{G.11}$$

Exploiting eq. (G.5) we obtain

$$ABe_i = -Af_i = -\widetilde{M}_{ji}e_j = BAe_i = M^*_{ji}e_j \tag{G.12}$$

and therefore

$$\widetilde{M} = -M^*, \qquad M^2 = -\lambda^* \mathbf{1}_{m_\lambda}. \tag{G.13}$$

Now we perform a unitary basis transformation on $\mathcal{V}(\lambda)$

$$e'_i = z_{ji}e_j \qquad \text{and therefore} \qquad f'_i \equiv -Be'_i = z^*_{ji}f_j \tag{G.14}$$

and explore its effect on M:

$$Ae'_i = (z^T M z^*)_{ji} f'_j, \qquad Af'_i = -(z^T M z^*)^*_{ji} e'_j. \tag{G.15}$$

Unitarity of M allows to choose a z such that

$$z^T M z^* = \text{diag}\ (e^{i\Theta_1}, \ldots, e^{i\Theta_{m_\lambda}}). \tag{G.16}$$

With $-\lambda^* \equiv \mu^2 = e^{2i\Theta_i}$ eq. (G.13), $e^{i\Theta_i} = \varepsilon_i \mu$, $\varepsilon_i = \pm 1$ and $\widehat{\varepsilon} = \text{diag}\ (\varepsilon_1, \ldots, \varepsilon_{m_\lambda})$ we finally see that A and B are represented by

$$A \to \begin{pmatrix} 0 & -\mu^* \widehat{\varepsilon} \\ \mu \widehat{\varepsilon} & 0 \end{pmatrix} K, \qquad B \to J_{m_\lambda} K \tag{G.17}$$

in the basis $\{e'_1, \ldots, e'_{m_\lambda}, f'_1, \ldots, f'_{m_\lambda}\}$.

b) $\mathcal{V} = \mathcal{V}_+$

One can easily prove that it is possible to find a vector $e_1 \in \mathcal{V}_+$ with $Ae_1 = e_1$ as a consequence of $A^2 = \mathbf{1}_{m_+}$. Then $f_1 \equiv -Be_1$ is orthogonal to e_1, $Af_1 = f_1$ and the space orthogonal to $\{e_1, f_1\}$ is invariant under A, B. Therefore we can repeat the previous steps in $\{e_1, f_1\}^\perp$ and continue the process until no dimension is left in $\mathcal{V}_+$. Thus we find that m_+ is even and

$$A \to \mathbf{1}_{m_+} K, \qquad B \to J_{m_+/2} K \tag{G.18}$$

in the basis $\{e_1, \ldots, e_{m_+/2}, f_1, \ldots, f_{m_+/2}\}$.

c) $\mathcal{V} = \mathcal{V}_-$

Now we have $A^2 = -\mathbf{1}_{m_-}$ and AB is a unitary operator with $(AB)^2 = \mathbf{1}_{m_-}$. Therefore we can find an eigenvector e_1 of AB with eigenvalue $\eta_1 = \pm 1$. We define $f_1 \equiv -Be_1$. Then

$$\langle Ae_1 | f_1 \rangle = \langle e_1 | ABe_1 \rangle^* = \eta_1 \tag{G.19}$$

and therefore

$$Ae_1 = \eta_1 f_1. \tag{G.20}$$

With the analogous basis as for eq. (G.18) we find that

$$A \to \begin{pmatrix} 0 & -\widehat{\eta} \\ \widehat{\eta} & 0 \end{pmatrix} K, \qquad \widehat{\eta} = \text{diag}\ (\eta_1, \ldots, \eta_{m_-/2}), \qquad B \to J_{m_-/2} K. \tag{G.21}$$

We have seen that the matrices for B in eqs. (G.17), (G.18) and (G.21) all have the form J_m eq. (G.3) and A is represented by matrices of the type (8.15). In general, with all cases a, b, c involved, trivial basis permutations lead to eq. (8.15). $\square$

H How to solve the generalized CP conditions

The methods and strategies to solve eq. (8.10) in the bases (8.11), (8.13), (8.14) and (8.15) will be discussed in this appendix. Thereby the ranges of the angles $0 < \Theta,\ \Theta' \leq \pi/2$ and $0 < \Theta_H < \pi$ have to be kept in mind because they will play a crucial rôle in the following.

The first observation is that the matrix $O(\vartheta)$ (8.12) has eigenvalues $\exp(\pm i\vartheta)$:

$$O(\vartheta)\begin{pmatrix} 1 \\ \pm i \end{pmatrix} = e^{\pm i\vartheta}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}. \tag{H.1}$$

If $0 < \vartheta < \pi$, there are no real eigenvalues. In the cases 1b) and 2a) eigenvectors of such a rotation matrix with eigenvalues ± 1 are required which do not exist in the ranges of Θ and Θ_H. Therefore, we have only the zero solutions.

Case 1c) is trivially solved.

Expressing A_2 by A_1 in case 2b) gives

$$A_2 = \frac{A_1 \cos\Theta_H - O(\Theta)A_1^*}{\sin\Theta_H}. \tag{H.2}$$

For the second equality gives the condition

$$\cos\Theta A_1 = \cos\Theta_H A_1^*. \tag{H.3}$$

Therefore, $A_1 = A_2 = 0$ for $\Theta \notin \{\Theta_H, \pi - \Theta_H\}$. Taking $\Theta = \Theta_H \neq \pi/2$, we necessarily have $A_1 \in \mathbf{R}^2$, $\Theta = \pi - \Theta_H \neq \pi/2$ requires A_1 to be imaginary. For $\Theta = \Theta_H = \pi/2$ eq. (H.3) gives no restriction and $A_1 \in \mathbf{C}^2$. A_2 is determined by eq. (H.2).

Case 2c) requires

$$A_2 = -dO(\Theta)A_1^* \qquad \text{and} \qquad O(2\Theta)A_1 = -d^2 A_1. \tag{H.4}$$

Thus $-d^2$ must be an eigenvalue of $O(2\Theta)$ of the form $e^{\pm 2i\Theta}$ or $A_1 = A_2 = 0$ if $d^2 \neq -e^{\pm 2i\Theta}$. For $\Theta = \pi/2$ and $d = \varepsilon$ the matrix A_1 is arbitrary. On the other hand, $\Theta < \pi/2$ requires A_1 to be an eigenvector of $O(2\Theta)$ from which the solution follows.

For the discussion of the remaining three cases we make the ansatz

$$A_1 \text{ (or } A) = \sum_{r,s=\pm} A_{rs} v_r v_s^T \qquad \text{with} \qquad v_\pm = \begin{pmatrix} 1 \\ \mp i \end{pmatrix}$$

and

$$O(\Theta)^T v_\pm = e^{\pm i\Theta} v_\pm, \qquad v_\pm^T O(\Theta') = e^{\pm i\Theta'} v_\pm^T. \tag{H.5}$$

Furthermore, after summation we obtain

$$A_1(\text{or } A) = \begin{pmatrix} A_{++} + A_{--} + A_{+-} + A_{-+} & i(-A_{++} + A_{--} + A_{+-} - A_{-+}) \\ i(-A_{++} + A_{--} - A_{+-} + A_{-+}) & -A_{++} - A_{--} + A_{+-} + A_{-+} \end{pmatrix}. \tag{H.6}$$

Application of eq. (H.5) in case 3a) readily gives

$$\sum_{r,s}(A_{rs}e^{i(r\Theta+s\Theta')} - A^*_{-r-s})v_r v_s^T = 0. \tag{H.7}$$

Evaluation of this equation leads to

$$A_{++}(e^{2i(\Theta+\Theta')} - 1) = A_{+-}(e^{2i(\Theta-\Theta')} - 1) = 0. \tag{H.8}$$

The only possibility for $A_{++} \cdot A_{+-} \neq 0$ requires $\Theta = \Theta' = \pi/2$ and, consequently, $A_{--} = -A^*_{++}$ and $A_{-+} = A^*_{+-}$. Inserting this into eq. (H.6) gives the first subcase of the solutions 3a). Going on to $\Theta = \Theta' < \pi/2$ one gets $A_{++} = A_{--} = 0$ and $A_{-+} = A^*_{+-}$. Inspecting again eq. (H.6) one sees that A has the same form as before but now its elements are real. Finally, $\Theta \neq \Theta'$ gives $A_{+-} = 0$ and $\Theta + \Theta' < \pi$. Therefore also $A_{++} = 0$ and thus $A = 0$ results.

The last two cases are more complicated. In 3b) we get the following equation for A_1:

$$O(\Theta)^T A_1 O(\Theta') + O(\Theta) A_1 O(\Theta')^T = 2\cos\Theta_H A_1^*. \tag{H.9}$$

Using the ansatz (H.5) we straightforwardly arrive at

$$\sum_{r,s}(A_{rs}\cos(r\Theta + s\Theta') - A^*_{-r-s}\cos\Theta_H)v_r v_s^T = 0. \tag{H.10}$$

Furthermore, A_2 is expressed in terms of A_1 by

$$A_2 = \frac{A_1\cos\Theta_H - O(\Theta)A_1^* O(\Theta')^T}{\sin\Theta_H}. \tag{H.11}$$

Let us first discuss $\Theta_H = \pi/2$. Then $A_{+-} = A_{-+} = 0$ because $\cos(\Theta - \Theta') \neq 0$. On the other hand, it is clear that $A_{++}, A_{--} \neq 0$ is only possible if $\Theta + \Theta' = \pi/2$. Thus with the help of eq. (H.6) we obtain the A_1 of the first solution of 3b). Applying eq. (H.11) we get with $\Sigma = \text{diag}\ (1, -1)$

$$\begin{aligned} A_2 &= -O(\Theta)\begin{pmatrix} a & b \\ b & -a \end{pmatrix}^* O(\Theta')^T = -\Sigma O(-\Theta)\begin{pmatrix} a & b \\ -b & a \end{pmatrix}^* O(\Theta')^T \\ &= -\Sigma\begin{pmatrix} a & b \\ -b & a \end{pmatrix}^* O(-\Theta - \Theta') = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -b & a \\ a & b \end{pmatrix}^*. \end{aligned} \tag{H.12}$$

Now we turn to the discussion of $\Theta_H \neq \pi/2$. Our starting point is

$$A_{++}(\cos^2(\Theta + \Theta') - \cos^2\Theta_H) = A_{+-}(\cos^2(\Theta - \Theta') - \cos^2\Theta_H) = 0. \tag{H.13}$$

We can only have both A_{++} and A_{+-} non–zero at the same time if $\cos^2(\Theta+\Theta') = \cos^2(\Theta-\Theta')$. It is easy to check that in such a case either $\Theta = \pi/2$ or $\Theta' = \pi/2$ (in addition we have $\Theta \neq \Theta'$ to prevent $\Theta_H = \pi/2$). Let us first discuss $\Theta = \pi/2$ and $\Theta' < \pi/2$. Then $\cos(\Theta \pm \Theta') = \mp \sin\Theta'$, $\cos\Theta_H = \varepsilon \sin\Theta'$ ($\varepsilon = \pm 1$) and $A_{--} = -\varepsilon A_{++}^*$, $A_{-+} = \varepsilon A_{+-}^*$. With eq. (H.6) we recover A_1 of the second solution of 3b). Using eq. (H.9) to replace $A_1 \cos\Theta_H$ in eq. (H.11) we obtain

$$A_2 = \frac{1}{2\sin\Theta_H}(O(\Theta)^T A_1^* O(\Theta') - O(\Theta) A_1^* O(\Theta')^T). \tag{H.14}$$

This equation is generally valid in case 3b). Taking $\Theta = \pi/2$ eq. (H.14) simplifies to

$$A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A_1^*. \tag{H.15}$$

The subcase $\Theta' = \pi/2$, $\Theta < \pi/2$ is dealt with in an analogous way.

Finally, we consider $\Theta, \Theta' < \pi/2$. Under this condition A_{++} and A_{+-} cannot be non–zero at the same time. Thus assuming first $\cos\Theta_H = -\varepsilon\cos(\Theta+\Theta')$ we have $A_{+-} = A_{-+} = 0$ and $A_{--} = -\varepsilon A_{++}^*$. This gives A_1 of the fourth solution of 3b). Now we apply eq. (H.14) and derive

$$\begin{aligned} A_2 &= \frac{1}{2\sin\Theta_H}(\Sigma O(\Theta)(\Sigma A_1^*)O(\Theta') - \Sigma O(-\Theta)(\Sigma A_1^*)O(-\Theta')) \\ &= \frac{1}{2\sin\Theta_H} A_1^*(O(\Theta+\Theta') - O(-\Theta-\Theta')) = A_1^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \tag{H.16}$$

In this way we have obtained the complete fourth solution. For $\cos\Theta_H = \varepsilon\cos(\Theta-\Theta')$ one can apply the same procedure.

Finally, let us discuss case 3c). A_1 is determined by

$$A_1 = -d^2 O(2\Theta)^T A_1 O(2\Theta'). \tag{H.17}$$

Thus with the decomposition (H.5) we get

$$A_{rs}(d^2 e^{2i(r\Theta+s\Theta')} + 1) = 0 \qquad \forall\, r,s = \pm. \tag{H.18}$$

There is exactly one possibility to have an arbitrary A_1, namely $d = i\varepsilon$, $\Theta = \Theta' = \pi/2$. A_2 is always computed by

$$A_2 = d^* O(\Theta)^T A_1^* O(\Theta') \tag{H.19}$$

in case 3c).

Next we observe that it is possible to get $A_{++}, A_{--} \neq 0$ by $\Theta + \Theta' = \pi/2$, $d = \varepsilon$. In this case $A_{+-} = A_{-+} = 0$. The A_2 is given by

$$A_2 = \varepsilon\Sigma O(\Theta)\Sigma A_1^* O(\Theta') = \varepsilon A_1^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{H.20}$$

There is an analogous case with $A_{++} = A_{--} = 0$ and $A_{+-}, A_{-+} \neq 0$ characterized by $\Theta = \Theta' \neq \pi/2$ and $d = i\varepsilon$.

Allowing for $A_{++}, A_{+-} \neq 0$ the conditions

$$-\, d^2 e^{2i(\Theta+\Theta')} = -d^2 e^{2i(\Theta-\Theta')} = 1 \tag{H.21}$$

must be fulfilled. In this case $\Theta' = \pi/2$ and $d = \varepsilon e^{-i\Theta}$. Then eqs. (H.6) and (H.19) determine A_1 and A_2, respectively. There are three similar cases characterized by $A_{++}, A_{-+} \neq 0$, $A_{--}, A_{+-} \neq 0$ and $A_{--}, A_{-+} \neq 0$.

Finally we are left with cases where only one A_{rs} is non–zero. The discussion goes along the line presented here.

I On the case $D_\Lambda \sim D_\Lambda \circ \psi_d$ in $so(2\ell)$

The non–trivial diagram automorphism of D_ℓ ($\ell \geq 5$) corresponds to exchanging $\alpha_{\ell-1}$ and α_ℓ in the Dynkin diagram (fig. 1), for D_4 there are other ones in addition (see sect. 4). Then $D_\Lambda \sim D_\Lambda \circ \psi_d$ for irreps with highest weight Λ is valid if and only if $n_{\ell-1} = n_\ell$. Here we want to show that if this is the case and if D_Λ is given in a CP basis (see subsect. 5.2) then the unitary matrix W establishing the equivalence

$$W D_\Lambda W^\dagger = D_\Lambda \circ \psi_d \tag{I.1}$$

can be chosen real and symmetric (see subsect. 9.2 for the use of this result).

To proceed with the proof we first describe how irreps of the kind $n_{\ell-1} = n_\ell$ can be built up as tensor products of the defining fundamental irrep D_{Λ_1}. For simplicity of notation we write $D_j \equiv D_{\Lambda_j}$. Then one can show that [24]

$$\wedge^k D_1 \cong D_k \; (k = 1, \ldots, \ell-2) \quad \text{and} \quad \wedge^{\ell-1} D_1 \cong D_{\Lambda'} \quad \text{with} \quad \Lambda' = \Lambda_{\ell-1} + \Lambda_\ell. \tag{I.2}$$

The symbol $\wedge^k$ denotes the k–fold antisymmetric tensor product. Thus D_Λ with $\Lambda = n_1\Lambda_1 + \ldots + n_{\ell-2}\Lambda_{\ell-2} + n_\ell(\Lambda_{\ell-1} + \Lambda_\ell)$ can be obtained as the irrep with highest weight in

$$D_1^{\otimes n_1} \otimes \ldots \otimes D_{\ell-2}^{\otimes n_{\ell-2}} \otimes D_{\Lambda'}^{\otimes n_\ell} \tag{I.3}$$

where the superscript $\otimes n_j$ indicates the n_j–fold tensor product of D_j. D_Λ can therefore be constructed from tensor products of D_1.

The résumé of this consideration is that if one can show that $\widetilde{A} D_1 \widetilde{A}^\dagger = D_1 \circ \psi_d$ is achieved with a real symmetric matrix $\widetilde{A}$ for D_1 in a CP basis the W for D_Λ with $n_{\ell-1} = n_\ell$ is obtained by suitable tensor products of $\widetilde{A}$ conserving reality and symmetry. Also in these tensor products $D_\Lambda(e_\alpha)$ will be real and $D_\Lambda(-iH_j)$ imaginary and symmetric if this is so in D_1. Then by reversing the argument leading to eq. (5.13) and by using $D_\Lambda(e_\alpha)^\dagger = -D_\Lambda(e_{-\alpha})$ we easily see that D_Λ is given in a CP basis.

Let us now consider D_1 and show first that there is a real and symmetric matrix A with $A D_1 A^\dagger = D_1 \circ \psi_d$ where D_1 is given in the natural basis of $so(2\ell)$ with real antisymmetric

matrices. We will closely follow the discussion in app. G of ref. [16]. Using the matrices $\{M_{pq}|\, 1 \le p < q \le 2\ell\}$ (see eq. (C.1)) as a basis of $so(2\ell)$ or of its complexification D_ℓ then $\{h_j = M_{2j-1,2j}|\, j = 1, \ldots, \ell\}$ defines a basis of the CSA. It is shown in ref. [16] that with the functionals ε_j on the CSA defined by

$$\varepsilon_j(h_k) = i\delta_{jk} \qquad (j, k = 1, \ldots, \ell) \tag{I.4}$$

all basis elements e_α with positive roots are given by

$$\begin{aligned} e_{\varepsilon_j+\varepsilon_k} &= \frac{1}{\sqrt{16(\ell-1)}}(M_{2j,2k} - iM_{2j,2k-1} - iM_{2j-1,2k} - M_{2j-1,2k-1}) \\ e_{\varepsilon_j-\varepsilon_k} &= \frac{1}{\sqrt{16(\ell-1)}}(M_{2j,2k} + iM_{2j,2k-1} - iM_{2j-1,2k} + M_{2j-1,2k-1}) \end{aligned} \tag{I.5}$$

with $1 \le j < k \le \ell$. Therefore

$$\Delta_+ = \{\varepsilon_j + \varepsilon_k, \varepsilon_j - \varepsilon_k|\, 1 \le j < k \le \ell\} \tag{I.6}$$

and the simple roots may be defined as

$$\alpha_j = \begin{cases} \varepsilon_j - \varepsilon_{j+1}, & j = 1, 2, \ldots, \ell - 1 \\ \varepsilon_{\ell-1} + \varepsilon_\ell, & j = \ell. \end{cases} \tag{I.7}$$

Then one can calculate

$$h_{\alpha_j} = -\frac{i}{4(\ell-1)}(M_{2j-1,2j} - M_{2j+1,2j+2}) \qquad (j = 1, 2, \ldots, \ell - 1)$$

and

$$h_{\alpha_\ell} = -\frac{i}{4(\ell-1)}(M_{2\ell-3,2\ell-2} + M_{2\ell-1,2\ell}). \tag{I.8}$$

With

$$t \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{I.9}$$

we have

$$\begin{aligned} h_{\alpha_\ell} &= -\frac{i}{4(\ell-1)}\mathrm{diag}\ (0_{2\ell-4}, t, t), \\ h_{\alpha_{\ell-1}} &= -\frac{i}{4(\ell-1)}\mathrm{diag}\ (0_{2\ell-4}, t, -t), \\ h_{\alpha_{\ell-2}} &= -\frac{i}{4(\ell-1)}\mathrm{diag}\ (0_{2\ell-6}, t, -t, 0_2),\ \text{etc.} \end{aligned} \tag{I.10}$$

In eq. (I.10) the symbol diag means arranging the matrices along the diagonal and 0_m denotes the $m \times m$ zero matrix. The basis elements corresponding to simple roots are represented by

$$e_{\alpha_\ell} = \frac{1}{\sqrt{16(\ell-1)}}\ \mathrm{diag}\ (0_{2\ell-4}, F')$$

with

$$F' = \begin{pmatrix} 0 & 0 & -1 & -i \\ 0 & 0 & -i & 1 \\ 1 & i & 0 & 0 \\ i & -1 & 0 & 0 \end{pmatrix} \tag{I.11}$$

and, for $j \neq \ell$,

$$e_{\alpha_j} = \frac{1}{\sqrt{16(\ell-1)}} \text{ diag } (0_{2(j-1)}, F, 0_{2(\ell-1-j)})$$

with

$$F = \begin{pmatrix} 0 & 0 & 1 & -i \\ 0 & 0 & i & 1 \\ -1 & -i & 0 & 0 \\ i & -1 & 0 & 0 \end{pmatrix}. \tag{I.12}$$

The diagram automorphism ψ_d is already uniquely determined by its action on h_{α_j} and e_{α_j}:

$$\psi_d(h_{\alpha_j}) = \begin{cases} h_{\alpha_j}, & j \leq \ell - 2 \\ h_{\alpha_\ell}, & j = \ell - 1, \\ h_{\alpha_{\ell-1}}, & j = \ell \end{cases} \quad \psi_d(e_{\alpha_j}) \text{ analogously} \tag{I.13}$$

Then one quickly confirms with eqs. (I.10), (I.11) and (I.12) that eq. (I.13) is reproduced by

$$\psi_d(X) = AXA \qquad \text{with} \qquad A = \text{diag } (\mathbf{1}_{2\ell-2}, -1, 1) \tag{I.14}$$

for $X \in so(2\ell)$ or its complexification D_ℓ. Since $D_1(X) = X$ we have obtained

$$(D_1 \circ \psi_d)(X) = \psi_d(X) = AXA = AD_1(X)A^\dagger \tag{I.15}$$

with A real and symmetric.

In the last step we have to switch to a CP basis by conserving the above properties of A. Defining

$$T = \text{diag } (s, \ldots, s) \qquad \text{with} \qquad s = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \tag{I.16}$$

we get

$$T^\dagger h_j T = i \text{ diag } (0_{2(j-1)}, \tau_3, 0_{2(\ell-j)}) \tag{I.17}$$

with the Pauli matrix τ_3. Furthermore, one readily verifies that

$$(T^\dagger M_{pq} T)_{rs} = T^*_{pr} T_{qs} - T^*_{qr} T_{ps}. \tag{I.18}$$

This allows to check easily that

$$(T^\dagger M_{2j,2k} T)_{rs} \qquad \text{and} \qquad (T^\dagger M_{2j-1,2k-1} T)_{rs} \qquad \text{are real } \forall\, r, s$$

and

$$(T^\dagger M_{2j,2k-1} T)_{rs} \qquad \text{and} \qquad (T^\dagger M_{2j-1,2k} T)_{rs} \qquad \text{are imaginary } \forall\, r, s.$$

Therefore, looking at eq. (I.5) we immediately see that we have

$$T^\dagger e_{\varepsilon_j \pm \varepsilon_k} T \qquad \text{real } \forall\, j, k = 1, \ldots, \ell \text{ with } j < k. \tag{I.19}$$

Thus the matrix T achieves a basis transformation into a CP basis. Finally, with the Pauli matrix τ_1 we obtain

$$\widetilde{A} = T^\dagger A T = \operatorname{diag}\left(\mathbf{1}_{2\ell-2}, -\tau_1\right) \tag{I.20}$$

which is real and symmetric. This proves the claim made in the beginning of this appendix.

$\widetilde{\mathcal{L}}$	$\mathcal{L}_c$	Aut $(\mathcal{L}_c)$/Int $(\mathcal{L}_c)$	$\psi^{\triangle}$
A_1	$su(2)$	$\{e\}$	inner
A_ℓ $(\ell \geq 2)$	$su(\ell+1)$	$\mathbf{Z}_2$	outer
B_ℓ $(\ell \geq 2)$	$so(2\ell+1)$	$\{e\}$	inner
C_ℓ $(\ell \geq 3)$	$sp(2\ell)$	$\{e\}$	inner
D_4	$so(8)$	S_3	inner
D_ℓ $(\ell = 5, 7, 9, \ldots)$	$so(2\ell)$	$\mathbf{Z}_2$	outer
D_ℓ $(\ell = 6, 8, 10, \ldots)$	$so(2\ell)$	$\mathbf{Z}_2$	inner
E_6	cE_6	$\mathbf{Z}_2$	outer
E_7	cE_7	$\{e\}$	inner
E_8	cE_8	$\{e\}$	inner
F_4	cF_4	$\{e\}$	inner
G_2	cG_2	$\{e\}$	inner

Table 1: The complete list of simple complex Lie algebras with their compact real forms and the structure of their automorphism groups. We have also indicated in which algebras the contragredient automorphism $\psi^{\triangle}$ is inner ($\psi^{\triangle} \in$ Int $(\mathcal{L}_c)$) or outer ($\psi^{\triangle} \notin$ Int $(\mathcal{L}_c)$). S_3 denotes the group of permutations of $\{1, 2, 3\}$ and $sp(2\ell)$ is the Lie algebra corresponding to the compact symplectic group $Sp(2\ell) = \{U \in U(2\ell) |\, U^T J U = J\}$ with

$$J = \begin{pmatrix} 0 & \mathbf{1}_\ell \\ -\mathbf{1}_\ell & 0 \end{pmatrix}.$$

$$\begin{array}{rcl} so(3) & \cong & su(2) \\ sp(2) & \cong & su(2) \\ sp(4) & \cong & so(5) \\ so(2) & \cong & u(1) \\ so(4) & \cong & su(2) \oplus su(2) \\ so(6) & \cong & su(4) \end{array}$$

Table 2: Complete list of isomorphisms within the four series of classical compact algebras $su(N)$ $(N \geq 2)$, $so(N)$ $(N \geq 2)$ and $sp(N)$ $(N = 2, 4, 6, \ldots)$. Of all these algebras only $so(2)$ and $so(4)$ are not simple. The last two isomorphisms are proved in apps. C and D, respectively.

References

[1] T.D. Lee and C.N. Yang, *Phys. Rev.* **104** (1956) 254.

[2] E. Ambler, R.W. Hayward, D.D. Hoppes, R.R. Hudson and C.S. Wu, *Phys. Rev.* **105** (1957) 1413.

[3] J.H. Christenson, J.W. Cronin, V.L. Fitch and R. Turlay, *Phys. Rev. Lett.* **13** (1964) 138.

[4] S.L. Glashow, *Nucl. Phys.* **22** (1961) 579;
S. Weinberg, *Phys. Rev. Lett.* **19** (1967) 1264;
A. Salam, in "Proc. of 8th Nobel Symposium", Aspenäsgården, 1968, ed. N. Svartholm (Almqvist and Wiksell, Stockholm, 1968).

[5] M. Kobayashi and T. Maskawa, *Prog. Theor. Phys.* **49** (1973) 652.

[6] J.C. Pati and A. Salam, *Phys. Rev. D* **8** (1973) 1240;
H. Georgi and S.L. Glashow, *Phys. Rev. Lett.* **32** (1974) 438.

[7] R. Slansky, *Phys. Rep.* **79** (1981) 1.

[8] N.V. Smolyakov, *Theor. and Math. Phys.* **50** (1982) 225.

[9] P. Binetruy, M.K. Gaillard and Z. Kunszt, *Nucl. Phys. B* **144** (1978) 141.

[10] G. Ecker, W. Grimus and W. Konetschny, *Nucl. Phys. B* **191** (1981) 465.

[11] L. O'Raifeartaigh, *Phys. Rev. B* **139** (1965) 1050;
S. Coleman and J. Mandula, *Phys. Rev.* **159** (1967) 1251;
W. Grimus, *J. Phys. A* **26** (1993) L435.

[12] J.C. Pati and A. Salam, *Phys. Rev. D* **10** (1975) 275.

[13] R.N. Mohapatra and J.C. Pati, *Phys. Rev. D* **11** (1975) 566 and 2558.

[14] J. Schwinger, *Phys. Rev.* **82** (1951) 914;
G. Lüders, *Dansk Mat. Fys. Medd.* **28** (1954) 5;
W. Pauli, in "Niels Bohr and the Development of Physics", ed. W. Pauli (Pergamon Press, New York, 1955).

[15] J.D. Bjorken and S.D. Drell, "Relativistic Quantum Fields" (McGraw–Hill, New York, 1965).

[16] J.F. Cornwell, "Group Theory in Physics" (Academic Press, London, 1984).

[17] W. Grimus, in "Proc. of 4th Hellenic School on Elementary Particle Physics", Corfu, 1992, eds. E.N. Gazis et al. (National Technical University, Zografou Campus, Athens).

[18] M. Gell–Mann, *Phys. Rev.* **125** (1962) 1067.

[19] H. Georgi, in "Proc. of Particles and Fields", A.I.P., ed. C.E. Carlson (New York, 1975);
H. Fritzsch and P. Minkowski, *Ann. Phys.* **93** (1975) 193.

[20] H. Georgi, "Lie Algebras in Particle Physics" (Benjamin/Cummings, Menlo Park, 1982).

[21] G.G. Ross, "Grand Unified Theories" (Benjamin/Cummings, Menlo Park, 1985).

[22] R.N. Mohapatra, "Unification and Supersymmetry" (Springer, New York, 1986).

[23] E.S. Abers and B. Lee, *Phys. Rep. C* **9** (1973) 1.

[24] H. Samelson, "Notes on Lie Algebras" (Springer, New York, 1990).

[25] N. Jacobson, "Lie Algebras" (Interscience Publishers, New York, 1962).

[26] V.S. Varadarajan, "Lie Groups, Lie Algebras, and Their Representations" (Springer, New York, 1984).

[27] G. Ecker, W. Grimus and H. Neufeld, *Nucl. Phys. B* **247** (1984) 70.

[28] J. Bernabéu, G.C. Branco and M. Gronau, *Phys. Lett. B* **169** (1986) 243.

[29] G. Ecker, W. Grimus and H. Neufeld, *J. Phys. A* **20** (1987) L807.

[30] R. Shaw, "Linear Algebra and Group Representations" (Academic Press, London, 1982).

[31] M. Hamermesh, “Group Theory and its Application to Physical Problems” (Addison–Wesley, Reading, MA, 1962).

[32] G. Senjanović and R.N. Mohapatra, *Phys. Rev. D* **12** (1975) 1502.

[33] R.N. Mohapatra and G. Senjanović, *Phys. Rev. Lett.* **44** (1980) 912; *Phys. Rev. D* **23** (1981) 165.

[34] W. Grimus and H. Urbantke, “Antilinear Operators and Group Representations”, preprint UWThPh-1994-30.

[35] R. Slansky, in “Proc. of First Workshop on Grand Unification”, eds. P.H. Frampton, S.L. Glashow and A. Yildiz (Math. Sci. Press, Brookline, 1980).

[36] T.D. Lee, *Phys. Rev. D* **8** (1973) 1226.

[37] P. Fayet, *Nucl. Phys. B* **78** (1974) 14.

[38] G. Ecker, W. Grimus and H. Neufeld, preprint TH.3780–CERN (1983) (this is a longer version of ref. [27]).

[39] W. Grimus, *Fortschr. Phys.* **36** (1988) 201.

[40] G.C. Branco, J.–M. Gérard and W. Grimus, *Phys. Lett. B* **136** (1984) 383.

[41] B.G. Wybourne, “Classical Groups for Physicists” (Wiley–Interscience, New York, 1974).

[42] R.N. Cahn, “Semi–Simple Lie Algebras and Their Representations” (Benjamin/ Cummings, Menlo Park, 1984).

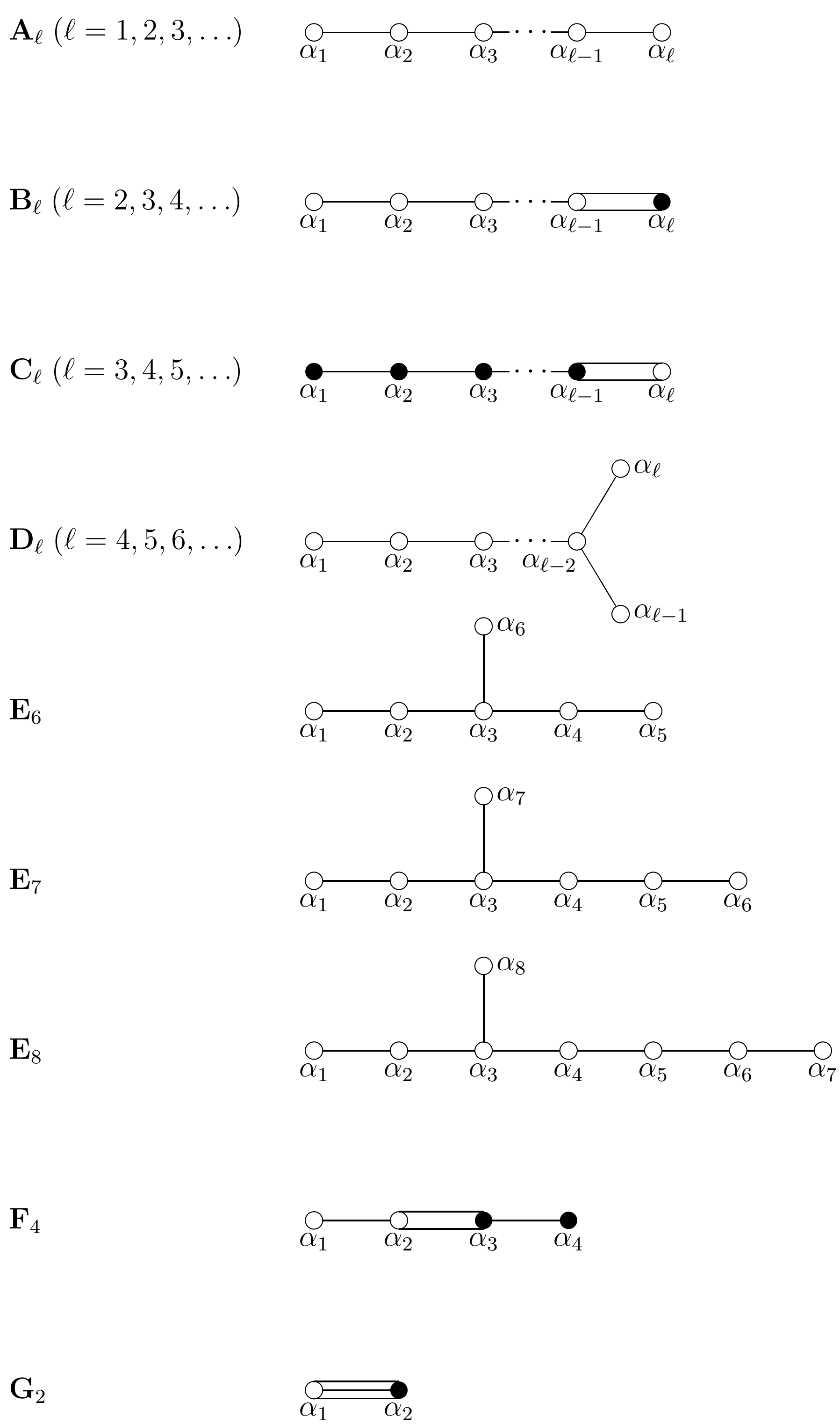

Figure 1: The Dynkin diagrams of all simple complex Lie algebras.